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Classical Mechanics: The Notes[1]

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Abstract: These notes cover analytical mechanics, the final level of classical mechanics for undergraduates. The Big Three core topics for this course are Rigid Body Motion, Normal Modes for Coupled Oscillations and Lagrangian and Hamiltonian Mechanics. To these I will add other topics, like fluid mechanics and classical fields (and possibly chaotic motion, time permitting). Students are assumed to have had an exposure to classical mechanics beyond the first-year level, such as seen in Physics 2E03. I will make an effort to keep the discussion self-contained by briefly reviewing the needed background material.

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1 Preliminaries

Classical Mechanics is the first physics topic most people meet and it is typically re-encountered several times within an undergraduate physics program. This repetition is not at all silly because it is a vast subject that rewards repeated study with deeper insight. Its modern importance is not diminished by the extension of classical methods to include relativity or by the ultimate replacement of the classical world-view by quantum mechanics. Besides its unique historical role in the development of the scientific method, classical tools also remain extremely useful as approximations for quantum systems, especially in regimes much larger than atomic sizes.

The purpose of these notes is to describe the calculational tools needed to analyze the behaviour of classical systems, assuming some familiarity with basic mechanics (which is also briefly reviewed in this section). The hope is to provide sufficient background to allow the reader to explore on their own the many more advanced topics to which these tools lead – like classical fields, continuum mechanics and chaotic behaviour – some of which are briefly touched on here.

1.1 Newton's Laws for Point Particles

This section starts with a quick reminder of Newton's Laws of motion as applied to point particles. A 'point particle' here is meant to be an object whose size is small enough to be in practice negligible and so whose properties are completely specified by giving its position as a function of time: $\mathbf{r}(t)$. We come back in later sections to making the notion of point particles more precise but for the time being we can think of point particles being the atoms from which larger objects are built. Much of the story of Classical Mechanics aims to clarify what can be said about the motion of macroscopic objects given that their microscopic constituents satisfy Newton's laws of motion.

The position vector, $\mathbf{r}(t)$, of a point particle gives the distance and direction to the particle at time t, measured relative to an origin of coordinates O, as shown Figure 1.

Referred to a right-handed Cartesian basis of orthogonal unit vectors, $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, where (for $i, j = x, y, z$)

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} := \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{1.1.1}$$

(which also defines the Kronecker delta symbol, δ_{ij}). The trajectory $\mathbf{r}(t)$ can be equivalently expressed in terms of its components: $x(t) = \mathbf{e}_x \cdot \mathbf{r}(t)$, $y(t) = \mathbf{e}_y \cdot \mathbf{r}(t)$ and $z(t) = \mathbf{e}_z \cdot \mathbf{r}(t)$,

$$\mathbf{r}(t) = x(t)\, \mathbf{e}_x + y(t)\, \mathbf{e}_y + z(t)\, \mathbf{e}_z \,, \tag{1.1.2}$$

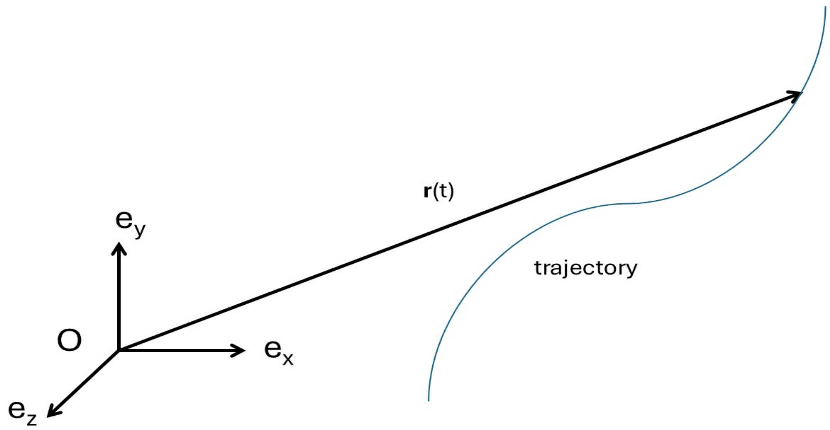


Figure 1. A sketch of the vector $\mathbf{r}(t)$ describing the trajectory of a point particle relative to the origin O of coordinates and a basis of unit vectors $\mathbf{e}_i$.

which in turn is often simply represented as a column vector

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} . \tag{1.1.3}$$

The velocity, $\mathbf{v}(t) := \dot{\mathbf{r}}(t) = \mathrm{d}\mathbf{r}/\mathrm{d}t$ of the trajectory is found by differentiation with respect to time and the acceleration is similarly defined as the derivative of the velocity $\mathbf{a}(t) := \dot{\mathbf{v}}(t) = \ddot{\mathbf{r}}(t) = \mathrm{d}^2\mathbf{r}/\mathrm{d}t^2$. Since the Cartesian basis vectors $\mathbf{e}_i$ are themselves time-independent these definitions imply the velocity has components $\mathbf{v} = v_x\,\mathbf{e}_x + v_y\,\mathbf{e}_y + v_z\,\mathbf{e}_z$ where $v_x(t) = \dot{x}(t)$, $v_y(t) = \dot{y}(t)$ and $v_z(t) = \dot{z}(t)$, and similarly for $\mathbf{a} = a_x\,\mathbf{e}_x + a_y\,\mathbf{e}_y + a_z\,\mathbf{e}_z$, so

$$\mathbf{v}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} \quad \text{and} \quad \mathbf{a}(t) = \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \end{pmatrix} . \tag{1.1.4}$$

Newton's first and second laws then state that there exists a family of reference frames – called *inertial frames* – for which a point-particle's trajectory is given by

$$m\mathbf{a} = \mathbf{F} \tag{1.1.5}$$

where m is the particle's *inertial mass* (which is an intrinisic characteristic of each particle) and $\mathbf{F}$ is the net force that is applied to the particle. Much of the content of Newton's Laws comes only after the forces summed to get $\mathbf{F}(t)$ are specified – perhaps as a function of $\mathbf{r}(t)$ and $\mathbf{v}(t)$ – along the trajectory.

1.1.1 Integration of the equations of motion

If the forces are given as explicit functions of time then (1.1.5) becomes a differential equation to be solved for $\mathbf{r}(t)$. For instance if $\mathbf{F}$ is independent of time then integrating (1.1.5) gives

$$\mathbf{v}(t) = \mathbf{v}_0 + \frac{\mathbf{F}}{m}(t-t_0) \quad \text{and} \quad \mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0(t-t_0) + \frac{\mathbf{F}}{2m}(t-t_0)^2\,, \tag{1.1.6}$$

where $\mathbf{v}_0$ and $\mathbf{r}_0$ are integration constants whose numerical values are determined in terms of the initial conditions: $\mathbf{v}(t_0) = \mathbf{v}_0$ and $\mathbf{r}(t_0) = \mathbf{r}_0$. This simple motion is a good approximation to the motion of a particle falling near the surface of the Earth if we take $\mathbf{F} = m\mathbf{g}$ where $\mathbf{g}$ is a universal constant acceleration whose value is 9.8 m/s^2 directed towards the Earth's centre. Because the basis vectors $\mathbf{e}_i$ are time-independent the integrations leading to (1.1.6) can be done component by component:

$$\mathbf{v}(t) = \begin{pmatrix} v_{x0} + F_x(t-t_0)/m \\ v_{y0} + F_y(t-t_0)/m \\ v_{z0} + F_z(t-t_0)/m \end{pmatrix} \quad \text{and} \quad \mathbf{r}(t) = \begin{pmatrix} x_0 + v_{x0}(t-t_0) + \frac{1}{2}F_x(t-t_0)^2/m \\ y_0 + v_{y0}(t-t_0) + \frac{1}{2}F_y(t-t_0)^2/m \\ z_0 + v_{z0}(t-t_0) + \frac{1}{2}F_z(t-t_0)^2/m \end{pmatrix}. \tag{1.1.7}$$

More commonly forces are specified as a function of position rather than (or as well as) being functions of time. Position-dependence introduces time-dependence as the positions move, but the complication comes because the time-dependence of the positions must be solved for as part of the determination of the time-dependence of the forces. In this case general solutions are only known for specific types of position-dependence.

1D Simple Harmonic Motion

Perhaps the simplest example for which motion in the presence of position-dependent forces can be solved in some generality is the case where the forces are linear functions of the positions. We consider here the simplest example of this type: the *simple harmonic oscillator* in one dimension.

The simple harmonic oscillator is defined by one-dimensional motion in the presence of a linear restoring force about an equilibrium position. In one dimension position is determined by a single number, so a linear restoring force has the form $F = -k(x - x_0)$ where the equilibrium position, $x = x_0$, is the place where F vanishes. Newton's 2nd law for such a problem is $m\ddot{x} = F$, leading to the differential equation for $y = x - x_0$:

$$\ddot{y} + \omega^2 y = 0 \quad \text{with} \quad \omega^2 = \frac{k}{m}\,. \tag{1.1.8}$$

This famously has general solution

$$y(t) = A\,\cos(\omega t + \delta)\,, \tag{1.1.9}$$

where A and δ are integration constants. Equivalently, y can be taken to be the real (or imaginary) part of a complex solution

$$y(t) = A_+ e^{i\omega t} + A_- e^{-i\omega t}\,, \tag{1.1.10}$$

where $A_\pm$ are integration constants. These describe sinusoidal oscillations with amplitude A, angular frequency ω and initial phase δ.

The force $F = -ky$ is conservative, coming from a potential energy $V = \frac{1}{2}\,ky^2$, and so the energy in such an oscillation is

$$E = \tfrac{1}{2} m\dot{y}^2 + \tfrac{1}{2} ky^2 = \tfrac{1}{2} mA^2\omega^2\,. \tag{1.1.11}$$

Damped oscillations

More complicated variations on the theme include *damped* harmonic oscillators, for which (1.1.8) is replaced by

$$\ddot{y} + \gamma\dot{y} + \omega^2 y = 0\,, \tag{1.1.12}$$

which has general solution

$$y(t) = A\,e^{-\gamma t}\cos(\omega t + \delta)\,, \tag{1.1.13}$$

or equivalently

$$y(t) = A_+ e^{(-\gamma + i\omega)t} + A_- e^{(-\gamma - i\omega)t}\,. \tag{1.1.14}$$

The main new ingredient here is the damping of the oscillation amplitude over time. Because the force $F = -\lambda\dot{y}$ is not conservative the energy

$$E = \tfrac{1}{2} m\dot{y}^2 + \tfrac{1}{2} ky^2 = \tfrac{1}{4} mA^2 e^{-2\lambda t}\Big\{\lambda^2 + 2\omega^2 + \lambda^2\cos[2(\omega t + \delta)] + 2\lambda\omega\sin[2(\omega t + \delta)]\Big\} \tag{1.1.15}$$

is not time-independent for $\lambda \neq 0$.

Forced oscillations

Another important variation is the *forced* damped harmonic oscillator, for which (1.1.8) becomes

$$\ddot{y} + \gamma\dot{y} + \omega^2 y = f(t) \tag{1.1.16}$$

where $f(t)$ is a specified function of time. This has general solution

$$y(t) = A\,e^{-\gamma t}\cos(\omega t + \delta) + y_p(t) \tag{1.1.17}$$

with

$$y_p(t) = \int_{-\infty}^{\infty} \mathrm{d}\mu\;\tilde{y}(\mu)\,e^{i\mu t} \quad \text{where} \quad \tilde{y}(\mu) = \frac{\tilde{f}(\mu)}{-\mu^2 + i\gamma\mu + \omega^2}\,, \tag{1.1.18}$$

where

$$\tilde{f}(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t\,f(t)\,e^{-i\mu t}\,. \tag{1.1.19}$$

For instance if $f(t) = f_0 \sin(\Omega t)$ then

$$\tilde{f}(\mu) = \left(\frac{f_0}{2i}\right) \delta(\mu - \Omega) - \left(\frac{f_0}{2i}\right) \delta(\mu + \Omega) \,, \tag{1.1.20}$$

and so

$$\begin{aligned} y_p(t) &= \frac{f_0}{2i(\Omega^2 + i\gamma\Omega - \omega^2)}\, e^{i\Omega t} - \frac{f_0}{2i(\Omega^2 - i\gamma\Omega - \omega^2)}\, e^{-i\Omega t} \\ &= \frac{f_0}{(\Omega^2 - \omega^2)^2 + \gamma^2\Omega^2} \Big[(\Omega^2 - \omega^2) \sin(\Omega t) - \gamma\Omega \cos(\Omega t) \Big] \,. \end{aligned} \tag{1.1.21}$$

In addition to oscillations at the specific frequency ω, the forced oscillator also oscillates with any frequency contained within the time-dependent driving force, with the oscillations at these other frequencies having a specific amplitude and phase. In particular the amplitude grows the closer the driving frequency is to the oscillator's natural frequency ω (the phenomenon known as resonance).

1.2 Two-body Problem

A great body of experience can be summarized by the statement that the origins of forces acting on a point particle come from the presence of other particles. This section explores the two-body problem partially to illustrate this but also partially to illustrate the cumbersomeness of changing variables in the traditional vector formulation of mechanics.

To this end consider Newton's laws as applied to two particles, which both exert a force on each other and are pushed by an external force. The expression of Newton's law (1.1.5) to these two particles is

$$m_1 \ddot{\mathbf{r}}_1 = \mathbf{F}_{12} + \mathbf{F}_1^{\rm ext} \quad \text{and} \quad m_2 \ddot{\mathbf{r}}_2 = \mathbf{F}_{21} + \mathbf{F}_2^{\rm ext} \,, \tag{1.2.1}$$

where $\mathbf{F}_{ab}$ denotes the force exerted on particle 'a' by particle 'b' and $\mathbf{F}_a^{\rm ext}$ denotes the external force acting on particle 'a'.

The presence of two bodies acting on one another provides an opportunity to state the third of Newton's laws, which applies to forces acting between two bodies. It states

$$\mathbf{F}_{ab} = -\mathbf{F}_{ba} \qquad \text{for all } a \text{ and } b \,. \tag{1.2.2}$$

In the present example this implies $\mathbf{F}_{12} = -\mathbf{F}_{21}$. As a result the forces acting between the two bodies cancel out if the two equations in (1.2.1) are added to one another:

$$m_1 \ddot{\mathbf{r}}_1 + m_2 \ddot{\mathbf{r}}_2 = \mathbf{F}_1^{\rm ext} + \mathbf{F}_2^{\rm ext} = \mathbf{F}_{\rm tot}^{\rm ext} \,. \tag{1.2.3}$$

This is more suggestively written to look like (1.1.5) if it is written

$$M \ddot{\mathbf{R}} = \mathbf{F}_{\rm tot}^{\rm ext} \,, \tag{1.2.4}$$

where

$$M := m_1 + m_2 \tag{1.2.5}$$

is the total mass of the two particles and the *centre of mass* coordinate $\mathbf{R}$ is defined by

$$\mathbf{R} := \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}\,. \tag{1.2.6}$$

In particular, in the absence of net external forces (1.2.4) expresses the conservation of centre-of-mass momentum:

$$\dot{\mathbf{P}} = 0 \quad \text{if } \mathbf{F}^{\rm ext}_{\rm tot} = 0 \quad \text{where} \quad \mathbf{P} := M\dot{\mathbf{R}} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2\,. \tag{1.2.7}$$

It is the dynamics of the relative inter-particle separation $\mathbf{r} := \mathbf{r}_1 - \mathbf{r}_2$ that is sensitive to the inter-particle forces. Taking the difference of the two equations in (1.2.1) and using the Third Law (1.2.2) implies

$$\mu\ddot{\mathbf{r}} = \mathbf{F}_{12} + \frac{m_2\mathbf{F}^{\rm ext}_1 - m_1\mathbf{F}^{\rm ext}_2}{M}\,, \tag{1.2.8}$$

where the *reduced mass* μ is defined by $\mu^{-1} := m_1^{-1} + m_2^{-1}$, or

$$\mu = \frac{m_1 m_2}{M}\,. \tag{1.2.9}$$

Notice the dependence on the external force cancels out in the special case of a constant gravitational field, for which $\mathbf{F}^{\rm ext}_a = m_a\mathbf{g}$ (for constant $\mathbf{g}$), leaving the evolution of $\mathbf{r}$ completely determined by the size of $\mathbf{F}_{12}$. In the event that the right-hand side of eqs. (1.2.4) depends only on $\mathbf{R}$ and the right-hand side of (1.2.8) depends only of $\mathbf{r}$ then eqs. (1.2.4) and (1.2.8) decouple and the evolution of $\mathbf{R}$ and $\mathbf{r}$ can be computed independently of one another.

This is true in particular when both $\mathbf{F}^{\rm ext}_a = m_a\mathbf{g}$ and $\mathbf{F}_{12}$ is a function only of $\mathbf{r}$. With these choices eq. (1.2.4) for $\mathbf{R}$ becomes $\ddot{\mathbf{R}} = \mathbf{g}$ and so integrates to give

$$\mathbf{R}(t) = \mathbf{R}_0 + \mathbf{V}_0(t - t_0) + \tfrac{1}{2}\mathbf{g}(t - t_0)^2\,, \tag{1.2.10}$$

as appropriate for a freely falling centre of mass. Here $\mathbf{V} = \dot{\mathbf{R}}$ denotes the centre-of-mass velocity and $\mathbf{V}_0 = \mathbf{V}(t_0)$ is its initial value.

1.2.1 Two Body Central Forces

Consider next the special case where $\mathbf{F}^{\rm ext}_a = m_a\mathbf{g}$ and $\mathbf{F}_{12} = -\nabla V(r)$ for some function V that depends only of the magnitude $r := |\mathbf{r}| = |\mathbf{r}_1 - \mathbf{r}_2|$. Under these circumstances eq. (1.2.8) becomes

$$\mu\ddot{\mathbf{r}} = -\nabla V(r) = -V'(r)\,\frac{\mathbf{r}}{r} = -V'(r)\,\mathbf{e}_r\,, \tag{1.2.11}$$

where $\mathbf{e}_r(t) := \mathbf{r}(t)/r(t)$ denotes the time-dependent unit radial vector that points instantaneously to particle 1 from particle 2.

In this case it is also convenient to decompose the left-hand side of (1.2.11) in terms of a basisi of unit vectors that are adapted to spherical coordinates built using the position vector $\mathbf{r}(t)$. Defining coordinates $\{r(t), \theta(t), \phi(t)\}$ in terms of $\{x(t), y(t), z(t)\}$ using

$$x = r\sin\theta\cos\phi\,, \quad y = r\sin\theta\sin\phi \quad \text{and} \quad z = r\cos\theta\,, \tag{1.2.12}$$

we see that

$$\mathbf{e}_r(t) = \frac{\mathbf{r}}{r} = \frac{x}{r}\,\mathbf{e}_x + \frac{y}{r}\,\mathbf{e}_y + \frac{z}{r}\,\mathbf{e}_z = \sin\theta\cos\phi\,\mathbf{e}_x + \sin\theta\sin\phi\,\mathbf{e}_y + \cos\theta\,\mathbf{e}_z\,. \tag{1.2.13}$$

It is convenient to define two linearly independent basis vectors by

$$\mathbf{e}_\theta := \frac{\partial\mathbf{e}_r}{\partial\theta} = \cos\theta\cos\phi\,\mathbf{e}_x + \cos\theta\sin\phi\,\mathbf{e}_y - \sin\theta\,\mathbf{e}_z\,, \tag{1.2.14}$$

and

$$\mathbf{e}_\phi := \frac{1}{\sin\theta}\frac{\partial\mathbf{e}_r}{\partial\phi} = -\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y\,. \tag{1.2.15}$$

As is easily checked the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ is also orthonormal: $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$.

Unlike for the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ however the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ is time-dependent because it is defined with the vector $\mathbf{e}_r$ parallel to $\mathbf{r}(t)$. As $x(t)$, $y(t)$ and $z(t)$ vary with the particle position so must also $r(t)$, $\theta(t)$ and $\phi(t)$ defined using (1.2.12). Explicitly

$$\begin{aligned}
\dot{\mathbf{e}}_r &= \dot\theta\,\mathbf{e}_\theta + \dot\phi\sin\theta\,\mathbf{e}_\phi \\
\dot{\mathbf{e}}_\theta &= \dot\theta\Big(-\sin\theta\cos\phi\,\mathbf{e}_x - \sin\theta\sin\phi\,\mathbf{e}_y - \cos\theta\,\mathbf{e}_z\Big) + \dot\phi\Big(-\cos\theta\sin\phi\,\mathbf{e}_x + \cos\theta\cos\phi\,\mathbf{e}_y\Big) \\
&= -\dot\theta\,\mathbf{e}_r + \dot\phi\cos\theta\,\mathbf{e}_\phi \\
\dot{\mathbf{e}}_\phi &= \dot\phi\Big(-\cos\phi\,\mathbf{e}_x - \sin\phi\,\mathbf{e}_y\Big) = -\dot\phi\Big(\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta\Big)\,.
\end{aligned} \tag{1.2.16}$$

In terms of this basis of spherical polar unit vectors we then have $\mathbf{r} = r\,\mathbf{e}_r$ and so

$$\dot{\mathbf{r}} = \dot r\,\mathbf{e}_r + r\,\dot{\mathbf{e}}_r = \dot r\,\mathbf{e}_r + r\dot\theta\,\mathbf{e}_\theta + r\sin\theta\dot\phi\,\mathbf{e}_\phi\,. \tag{1.2.17}$$

Differentiating again gives

$$\begin{aligned}
\ddot{\mathbf{r}} &= \ddot r\,\mathbf{e}_r + \dot r\,\dot{\mathbf{e}}_r + \dot r\dot\theta\,\mathbf{e}_\theta + r\ddot\theta\,\mathbf{e}_\theta + r\dot\theta\,\dot{\mathbf{e}}_\theta + \dot r\sin\theta\dot\phi\,\mathbf{e}_\phi + r\cos\theta\dot\theta\dot\phi\,\mathbf{e}_\phi + r\sin\theta\ddot\phi\,\mathbf{e}_\phi + r\sin\theta\dot\phi\,\dot{\mathbf{e}}_\phi \\
&= \Big(\ddot r - r\dot\theta^2 - r\sin^2\theta\dot\phi^2\Big)\mathbf{e}_r + \Big(2\dot r\dot\theta + r\ddot\theta - r\dot\phi^2\sin\theta\cos\theta\Big)\mathbf{e}_\theta \\
&\qquad + \Big(2\dot r\dot\phi\sin\theta + 2r\dot\theta\dot\phi\cos\theta + r\ddot\phi\sin\theta\Big)\mathbf{e}_\phi
\end{aligned} \tag{1.2.18}$$

Using this basis the three components of (1.2.11) give the following three differential equations to be solved for $r(t)$, $\theta(t)$ and $\phi(t)$:

$$\mu\Big(\ddot r - r\dot\theta^2 - r\sin^2\theta\dot\phi^2\Big) + V'(r) = 0 \tag{1.2.19a}$$

$$2\dot r\dot\theta + r\ddot\theta - r\dot\phi^2\sin\theta\cos\theta = 0 \tag{1.2.19b}$$

$$\text{and} \quad 2\dot r\dot\phi\sin\theta + 2r\dot\theta\dot\phi\cos\theta + r\ddot\phi\sin\theta = 0\,. \tag{1.2.19c}$$

To solve these it is convenient to adapt the spherical coordinates so that the particle initially satisfies $\theta(t_0) = \frac{\pi}{2}$ and $\dot{\theta}(t_0) = 0$. This amounts to choosing our polar coordinates so that the plane spanned by $\mathbf{r}_0$ and $\mathbf{v}_0$ is the equator. But when this is true (1.2.19b) implies $\ddot{\theta}$ must also vanish at $t = t_0$. This shows that if the motion is initially in the equatorial plane then it always remains there, with $\theta = \frac{\pi}{2}$ providing a solution to (1.2.19b) for all time.

With this choice for θ the remaining pieces of (1.2.19) become

$$\mu\Big(\ddot{r} - r\dot{\phi}^2\Big) + V'(r) = 0 \tag{1.2.20a}$$

$$\text{and} \quad 2\dot{r}\dot{\phi} + r\ddot{\phi} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\Big(r^2\dot{\phi}\Big) = 0\,. \tag{1.2.20b}$$

The second of these equations says that the time derivative of the combination $r^2\dot{\phi}$ vanishes and so is solved by choosing

$$r^2\dot{\phi} = J \tag{1.2.21}$$

where J is an integration constant. Using this in (1.2.20a) then allows it to be written

$$\mu\ddot{r} - \frac{\mu J^2}{r^3} + V'(r) = 0\,. \tag{1.2.22}$$

Multiplying this through by $\dot{r}$ allows it to be written as a total derivative, and so integrating gives

$$\tfrac{1}{2}\mu\dot{r}^2 + \frac{\mu J^2}{2r^2} + V(r) = E \tag{1.2.23}$$

for a new integration constant E.

The qualitative form for the solutions can be found by plotting the effective potential energy $V_{\rm eff} = (\mu J/2r^2) + V(r)$ appearing in (1.2.23) as a function of r (see Fig. 3), together with the value of E (see Fig. 2). In this type of plot the kinetic energy of the radial motion, $\frac{2}{1}\mu\dot{r}^2$, is proportional to the difference $E - V_{\rm eff}$ and so solutions exist only when this difference is positive. Regions with $E < V_{\rm eff}$ are *classically forbidden*. The origin $r = 0$ is always forbidden (for nonzero J) provided the potential is less singular there than $1/r^2$.

For motion described by the energy shown in Fig. 2 an initially decreasing $r(t)$ shrinks until it reaches the turning point after which it starts to grow without bound. This describes a 'scattering' trajectory (as opposed to a 'bound' trajectory for which r cannot escape beyond a finite maximum value – see Fig. 3). The scattering angle for such an encounter is computed by using (1.2.21) to evaluate

$$\Delta\phi = \phi(t \to \infty) - \phi(t \to -\infty)\,. \tag{1.2.24}$$

The problem is solved once (1.2.23) is integrated to give $r(t)$, as can be done by performing the following integral (a solution by 'quadratures'):

$$t - t_0 = \int_{r_0}^{r} \frac{\mathrm{d}r}{\dot{r}} = \int_{r_0}^{r} \frac{\mathrm{d}x}{\sqrt{2[E - V(x)]/\mu - (J^2/x^2)}}\,. \tag{1.2.25}$$

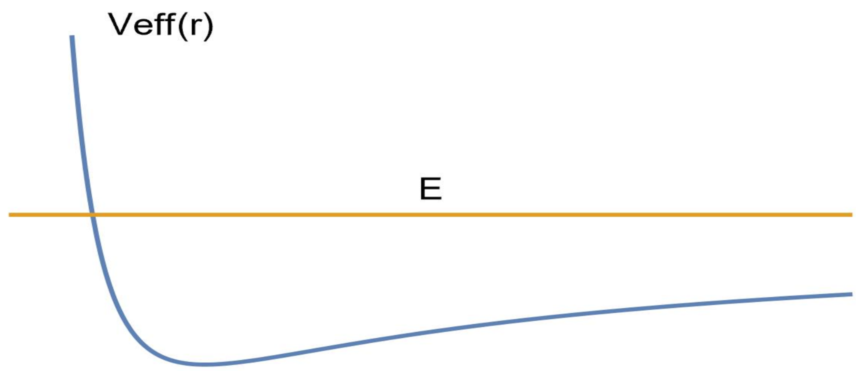


Figure 2. A sketch of $V_{\rm eff}(r) \propto -1/r$ vs r (with r increasing to the right) on which is superimposed a particular (positive) value of E. Classically allowed regions are those where $V_{\rm eff}(r) \le E$ and $\dot r = 0$ at the turning points where $V_{\rm eff}(r) = E$.

Once $r(t)$ is found then $\phi(t)$ is obtained by integrating (1.2.21). These integrals can be done quite explicitly for some choices of $V(r)$. Famously, in the special case where $V \propto 1/r$ these arguments lead to the conclusion that the orbits $r(\theta)$ traced out by the trajectories $r(t)$ and $\theta(t)$ are conic sections.

1.2.2 Kepler Problem

Perhaps the best-known example of the central body type is the *Kepler problem*, for which the central-force potential is assumed to be the one responsible for Newton's Law of Universal gravitation:

$$\mathbf{F}_{12} = -\frac{Gm_1m_2}{r^2}\,\mathbf{e}_r \quad \text{for which} \quad V(r) = -\frac{Gm_1m_2}{r}\,. \tag{1.2.26}$$

In this case the required integrations can be done explicitly and doing them allows Kepler's phenomenological orbital laws to be derived as consequences of Newton's second law (1.2.8) together with his law of gravitation (1.2.26).

In this case the radial equation (1.2.22) simplifies to

$$\ddot r - \frac{J^2}{r^3} + \frac{GM}{r^2} = 0\,, \tag{1.2.27}$$

once the definition $\mu = m_1m_2/M$ is used, where $M = m_1 + m_2$. Multiplying this through by $\dot r$ shows that the energy conservation equation then is

$$\tfrac12 \dot r^2 + \frac{J^2}{2r^2} - \frac{GM}{r} = \mathcal{E} \tag{1.2.28}$$

for a new integration constant $\mathcal{E}$.

The qualitative form for the solutions are again found by plotting the energy equation (1.2.23) as a function of r, as in Fig. 3. A negative value for E is also plotted in this figure, showing that when $E < 0$ the classically allowed region only includes a finite range of values $r_{\rm min} \le r \le r_{\rm max}$, as appropriate for a bound orbit.

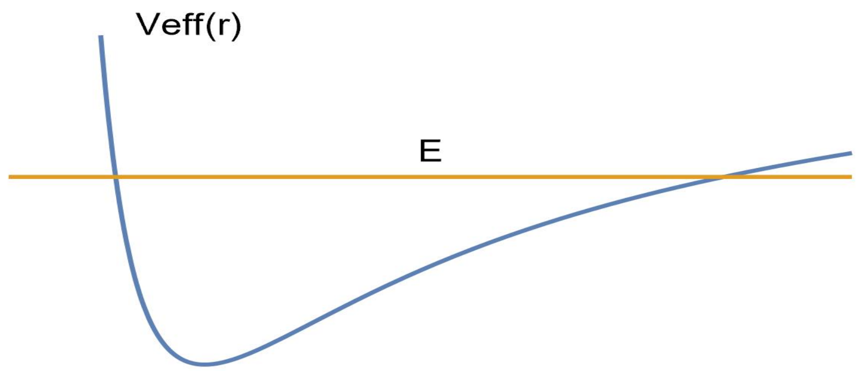


Figure 3. A sketch of $V_{\rm eff}(r) \propto -1/r$ vs r (with r increasing to the right) on which is superimposed a particular (negative) value of E. Classically allowed regions are those where $V_{\rm eff}(r) \le E$ and $\dot r = 0$ at the turning points where $V_{\rm eff}(r) = E$.

Circular orbits are the special case where r remains the same for all time, and so $r_{\rm min} = r_{\rm max}$ (and so $e = 0$ and so $a \to r_c$ becomes the circle's radius). The figure shows, circular orbits can only occur at radii r_c for which $V'_{\rm eff}(r_c) = 0$. As the figure also makes clear, circular orbits are the orbits with minimum energy for a given angular momentum J.

For the Keplerian potential (1.2.26) vanishing derivative implies

$$-\frac{J^2}{r_c^3} + \frac{GM}{r_c^2} = 0 \quad \text{and so} \quad r_c = \frac{J^2}{G(m_1+m_2)} \qquad \text{(circular)}\,. \tag{1.2.29}$$

Since this implies $GM/r_c = J^2/r_c^2$ it follows that the energy of such an orbit is

$$E = \frac{J^2}{2r_c^2} - \frac{GM}{r_c} = -\frac{J^2}{2r_c^2} = -\frac{GM}{2r_c} \qquad \text{(circular)}\,. \tag{1.2.30}$$

Eliminating the circle's radius from these gives the relation $E = E_c(J)$ that minimizes the energy for a given J:

$$E_c = -\frac{1}{2}\left(\frac{GM}{J}\right)^2 \qquad \text{(circular)}\,. \tag{1.2.31}$$

More generally the radial position changes as the orbiting objects move. For the Kepler problem the radius changes because bound orbits turn out to be ellipses with the orbitted particle at one focus (as we see below), so $r_{\rm min} = a(1-e)$ and $r_{\rm max} = a(1+e)$ are the points of closest and furthest approach, where a is the orbit's semi-major axis and $0 \le e \le 1$ is its eccentricity.

To see why bound orbits are ellipses (and unbound orbits are hyperbolae) we seek the shapes $r(\phi)$ of the orbits rather than $r(t)$ and $\phi(t)$ separately. These shapes are most easily found from the above equations by starting with (1.2.27) and (1.2.28) and changing variables to $r(\phi)$ using $\dot r = r'\dot\phi = r' J/r^2$ where primes here denote differentiation with respect to ϕ and $\dot\phi$ is eliminated using (1.2.21). Proceeding in this way (1.2.28) becomes

$$\tfrac12 \dot r^2 + \frac{J^2}{2r^2} - \frac{GM}{r} = \frac{(r')^2 J^2}{2r^4} + \frac{J^2}{2r^2} - \frac{GM}{r} = \left[\tfrac12 (u')^2 + \tfrac12 u^2 - \left(\frac{GM}{J^2}\right) u\right] J^2 = \mathcal{E} \tag{1.2.32}$$

where the last equality performs the change of variables $u = 1/r$. Eq. (1.2.27) similarly becomes

$$\ddot r - \frac{J^2}{r^3} + \frac{GM}{r^2} = \left[\frac{r''}{r^2} - \frac{2(r')^2}{r^3} - \frac{1}{r} + \frac{GM}{J^2}\right]\frac{J^2}{r^2} = \left[-u'' - u + \frac{GM}{J^2}\right] J^2 u^2 = 0\,. \tag{1.2.33}$$

The point of this exercise is the equations for $u(\phi)$ are simple to integrate since (1.2.33) is Newton's 2nd law for a harmonic oscillator that oscillates around $u = GM/J^2$ and (1.2.32) is the energy equation for such an oscillator. The general solution is therefore easy to write down:

$$u(\phi) = \frac{1}{r(\phi)} = \frac{GM}{J^2} + A\cos\Big(\phi + \phi_0\Big)\,, \tag{1.2.34}$$

where A and ϕ_0 are integration constants. This answer could also be found by changing variables from t to ϕ in the explicit quadrature in (1.2.25) and then performing the integral.

It is convenient to choose the origin of ϕ so that $r(\phi_0)$ is the point of closest approach of the orbiting bodies (the *periapsis*), which is the point where $u(\phi)$ is the largest. From (1.2.34) this implies $\phi_0 = 0$. Evaluating (1.2.32) using (1.2.34) allows the integration constant A to be related to the energy $\mathcal{E}$, with

$$\mathcal{E} = \left[\tfrac12 (u')^2 + \tfrac12 u^2 - \left(\frac{GM}{J^2}\right) u\right] J^2 = \tfrac12 \left[A^2 - \left(\frac{GM}{J^2}\right)^2\right] J^2 \tag{1.2.35}$$

and so

$$A^2 = \frac{2\mathcal{E}}{J^2} + \left(\frac{GM}{J^2}\right)^2\,. \tag{1.2.36}$$

Eq. (1.2.34) is to be compared with the equation of an ellipse in polar coordinates,

$$r(\phi) = \frac{a(1-e^2)}{1 + e\cos\phi}\,, \tag{1.2.37}$$

where again ϕ is chosen so that $\phi = 0$ is the point of closest approach. Here a is the semi-major axis and $0 \le e \le 1$ is the eccentricity. Comparing gives expressions for the conserved quantities $\mathcal{E}$ and J in terms of the orbital parameters a and e:

$$a(1-e^2) = \frac{J^2}{GM} \qquad \text{and} \qquad \frac{e}{a(1-e^2)} = A\,, \tag{1.2.38}$$

and so

$$J^2 = GMa(1-e^2) \tag{1.2.39}$$

and

$$\mathcal{E} = \tfrac{1}{2}\left[A^2 - \left(\frac{GM}{J^2}\right)^2\right]J^2 = \tfrac{1}{2}\left[\frac{e^2-1}{a^2(1-e^2)^2}\right]GMa(1-e^2) = -\frac{GM}{2a}\,. \tag{1.2.40}$$

An ellipse becomes a circle when the eccentricity vanishes, and notice that (1.2.38) implies $a \to J^2/GM$ in this limit (in agreement with (1.2.29)).

The period of the orbit is

$$\begin{aligned} T = \int_0^{2\pi} \frac{\mathrm{d}\phi}{\dot\phi} = \frac{1}{J}\int_0^{2\pi} \mathrm{d}\phi \; r^2(\phi) &= \frac{a(1-e^2)}{\sqrt{GMa(1-e^2)}} \int_0^{2\pi} \mathrm{d}\phi \frac{a^2(1-e^2)^2}{(1+e\cos\phi)^2} \\ &= \sqrt{\frac{a^3(1-e^2)^3}{GM}}\left[\frac{2\pi}{(1-e^2)^{3/2}}\right] = 2\pi\sqrt{\frac{a^3}{GM}} \end{aligned} \tag{1.2.41}$$

from which Kepler's third law (as improved by Newton) follows:

$$\frac{a^3}{T^2} = \frac{GM}{(2\pi)^2}\,. \tag{1.2.42}$$

1.2.3 Conservation Laws

Of course the ability to solve the above problems so explicitly relies on being able to perform so many time integrations. The ability to do so in the above discussion is not an accident, and can be traced to the presence of various *conservation laws*.

In particular, it is the assumption that $\mathbf{F}_{12} = -\nabla V$ for some choice for $V(\mathbf{r})$ that is responsible for energy conservation. To see why, notice that this assumption allows (1.2.11) to be written $\mu\ddot{\mathbf{r}} + \nabla V = 0$. Taking the dot product of this with $\dot{\mathbf{r}}$ then implies

$$\dot{\mathbf{r}}\cdot\Big(\mu\ddot{\mathbf{r}} + \nabla V\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\tfrac{1}{2}\mu\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + V\Big) = 0 \tag{1.2.43}$$

which expresses conservation of energy, E, for this system where

$$E = \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 + V(\mathbf{r}) = \frac{m_1 m_2}{2M}(\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2)^2 + V(\mathbf{r}) \tag{1.2.44}$$

is a constant along any solution $\mathbf{r}(t)$. As the particle moves there is a transfer from potential energy V to kinetic energy

$$K := \tfrac{1}{2}\,\mu\dot{\mathbf{r}}^2\,, \tag{1.2.45}$$

in such a way as to ensure the total energy $E = K + V$ remains unchanged.

Similarly, the assumption that $\mathbf{F}_{12}$ is parallel to $\mathbf{r}$ means $\mathbf{r}\times\mathbf{F}_{12} = 0$, and this implies a second conservation law. (In the above example this assumption is a consequence of choosing

$\mathbf{F}_{12} = -\nabla V$ where V depends only on the radial coordinate $r = |\mathbf{r}|$). The conservation law can be found by taking the cross product of (1.2.11) with $\mathbf{r}$:

$$\mathbf{r} \times \Big(\mu \ddot{\mathbf{r}} + \nabla V\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mu \mathbf{r} \times \dot{\mathbf{r}}\Big) = 0 \tag{1.2.46}$$

where the first equality uses $\mathbf{r} \times \mathbf{F}_{12} = 0$. Clearly the *angular momentum*

$$\mathbf{J} := \mathbf{r} \times (\mu \dot{\mathbf{r}}) \tag{1.2.47}$$

must be a constant vector along the entire trajectory $\mathbf{r}(t)$ for all t. Because $\mathbf{J}$ is always perpendicular to $\dot{\mathbf{r}}$ its conservation implies the motion takes place entirely within a plane in space that is perpendicular to $\mathbf{J}$. This is ultimately what allowed us to adapt our polar coordinates to ensure $\theta = \frac{\pi}{2}$ for all times. Eq. (1.2.21) then expresses that the magnitude $J = |\mathbf{J}|$ of the angular momentum is constant.

Conservation laws are clearly very useful when integrating the equations of motion. But how does one know when they exist? One of the points of the later Lagragian and Hamiltonian formulations of mechanics encountered in §2 and §7 is that they allow a very general answer to this question by identifying a deep connection between conservation laws and symmetries.

Before leaving the subject of conservation laws it is worth pointing out that there is more to a useful conservation law than just finding something that remains constant along a trajectory found by solving Newton's equations of motion. Indeed, for the two-body example just studied there are twelve conserved quantities that are preserved by the evolution (1.2.1) (compared to the seven quantities contained within $\mathbf{P}$, E and $\mathbf{J}$).

The six 'conserved' quantities can be taken to be the initial conditions $\mathbf{r}_{a0}$ and $\mathbf{v}_{a0}$ for the positions and velocities for the two particles. These are clearly conserved in the sense that every point along a trajectory $\mathbf{r}_a(t)$ shares the same values for these initial conditions, and in that sense the initial conditions can be regarded as independent of time along the trajectory.

These kinds of trivial conserved quantities differ from the useful ones in several important ways. First – unlike the expressions for E, $\mathbf{P}$ or $\mathbf{J}$ – the formula giving $\mathbf{r}_{a0}$ and $\mathbf{v}_{a0}$ as a function of $\mathbf{r}_a(t)$ and $\mathbf{v}(t)$ depends in a detailed way on t_0. Second, the expressions for $\mathbf{r}_{a0}$ and $\mathbf{v}_{a0}$ are not *additive* – unlike the expressions for energy, momentum or angular momentum – inasmuch as they do not arise as a sum over separate contributions coming from each of the two particles.

Although this additivity is explicit in the expression (1.2.7) for the conserved linear momentum, it only becomes true for energy and angular momentum once their definitions are extended to include the centre-of-mass motion. For instance, suppose that the external forces also have the property that $\mathbf{F}_a^{\mathrm{ext}} = -\nabla_a U$ for some function of position $U(\mathbf{r}_1, \mathbf{r}_2)$, where ∇_a denotes differentiation with respect to $\mathbf{r}_a$. In this case the equations of motion (1.2.1) for each body become $m_a\ddot{\mathbf{r}}_a + \nabla_a V + \nabla_a U = 0$ and so

$$\begin{aligned} 0 &= \dot{\mathbf{r}}_1 \cdot \Big(m_1 \ddot{\mathbf{r}}_1 + \nabla_1 V + \nabla_1 U \Big) + \dot{\mathbf{r}}_2 \cdot \Big(m_2 \ddot{\mathbf{r}}_2 + \nabla_2 V + \nabla_2 U \Big) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \Big(\tfrac12 m_1 \dot{\mathbf{r}}_1^2 + \tfrac12 m_2 \dot{\mathbf{r}}_2^2 + V + U \Big) \,, \end{aligned} \tag{1.2.48}$$

showing that the total conserved energy is indeed additive:

$$E = \tfrac12 m_1 \dot{\mathbf{r}}_1^2 + \tfrac12 m_2 \dot{\mathbf{r}}_2^2 + V + U = \tfrac12 M \dot{\mathbf{R}}^2 + \tfrac12 \mu \dot{\mathbf{r}}^2 + V + U \,, \tag{1.2.49}$$

where the last equality uses the definitions of $\mathbf{R}$, $\mathbf{r}$, M and μ.

A similar story goes through for angular momentum when $\mathbf{F}_a^{\rm ext} = 0$. In this case the equations of motion (1.2.1) imply $m_1 \ddot{\mathbf{r}}_1 - \mathbf{F}_{12} = m_2 \ddot{\mathbf{r}}_2 + \mathbf{F}_{12} = 0$ and so when $\mathbf{F}_{12}$ is parallel to $\mathbf{r}_1 - \mathbf{r}_2$ we have $(\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}_{12} = 0$ and so

$$0 = \mathbf{r}_1 \times \Big(m_1 \ddot{\mathbf{r}}_1 - \mathbf{F}_{12} \Big) + \mathbf{r}_2 \times \Big(m_2 \ddot{\mathbf{r}}_2 + \mathbf{F}_{12} \Big) = \frac{\mathrm{d}}{\mathrm{d}t} \Big[\mathbf{r}_1 \times (m_1 \dot{\mathbf{r}}_1) + \mathbf{r}_2 \times (m_2 \dot{\mathbf{r}}_2) \Big] \tag{1.2.50}$$

showing that the conserved angular momentum can be written in the additive form

$$\mathbf{J} = \mathbf{r}_1 \times (m_1 \dot{\mathbf{r}}_1) + \mathbf{r}_2 \times (m_2 \dot{\mathbf{r}}_2) \,. \tag{1.2.51}$$

What will be useful about the methods of §2 and §7 below is that they systematically lead to conserved quantities that are additive in this same way.

1.3 More is Similar

Before moving to alternative formulations of mechanics this section first extends some of the previous discussion to N particles. This proves to be useful in practice because it helps clarify how more complicated objects move under the assumption that their underlying constituents – perhaps the atoms from which they are made – behave as classical point particles.

1.3.1 Macro vs Micro

Suppose a macroscopic object, $\mathcal{O}$, is made up of a collection of N point-like atoms with the atoms labelled by an index $a, b = 1, \cdots, N$. These atoms experience external forces $\mathbf{F}_a^{\rm ext}$ and mutually interact through forces $\mathbf{F}_{ab}$, which as above describe the force acting on particle 'a' due to particle 'b'. Writing out Newton's 2nd law (1.1.5) for the motion of each atom then gives the system of equations

$$\begin{aligned} m_1 \, \ddot{\mathbf{r}}_1 &= \phantom{\mathbf{F}_{21}} + \mathbf{F}_{12} + \mathbf{F}_{13} + \cdots + \mathbf{F}_{1N} + \mathbf{F}_1^{\rm ext} \\ m_2 \, \ddot{\mathbf{r}}_2 &= \mathbf{F}_{21} \phantom{+ \mathbf{F}_{12}} + \mathbf{F}_{23} + \cdots + \mathbf{F}_{2N} + \mathbf{F}_2^{\rm ext} \\ m_3 \, \ddot{\mathbf{r}}_3 &= \mathbf{F}_{31} + \mathbf{F}_{32} + \phantom{\mathbf{F}_{33}} \cdots + \mathbf{F}_{3N} + \mathbf{F}_3^{\rm ext} \\ \vdots \;\; &= \\ m_N \, \ddot{\mathbf{r}}_N &= \mathbf{F}_{N1} + \mathbf{F}_{N2} + \mathbf{F}_{N3} + \cdots \phantom{+ \mathbf{F}_{NN}} + \mathbf{F}_N^{\rm ext} \,. \end{aligned} \tag{1.3.1}$$

The laws of motion for the entire macroscopic object must follow as consequences of eqs. (1.3.1), and at first sight it seems remarkable that any simple laws should be possible at all for macroscopic objects if this is so. A wonderful thing happens if all of these equations are added together, however, since then Newton's third law (which states that $\mathbf{F}_{ab} = -\mathbf{F}_{ba}$ for all a and b) implies that all of the $\mathbf{F}_{ab}$ cancel in the sum, leaving

$$m_1 \ddot{\mathbf{r}}_1 + m_2 \ddot{\mathbf{r}}_2 + \cdots + m_N \ddot{\mathbf{r}}_N = \mathbf{F}_1^{\rm ext} + \cdots + \mathbf{F}_N^{\rm ext} . \tag{1.3.2}$$

This takes the same form as did Newton's law for each atom:

$$M \ddot{\mathbf{R}} = \mathbf{F}_{\rm tot}^{\rm ext} , \tag{1.3.3}$$

with total mass and net external force given by

$$M := \sum_{a=1}^{N} m_a , \qquad \mathbf{F}_{\rm tot}^{\rm ext} := \sum_{a=1}^{N} \mathbf{F}_a^{\rm ext} , \tag{1.3.4}$$

provided one defines

$$\mathbf{R} := \frac{1}{M} \sum_{a=1}^{N} m_a \, \mathbf{r}_a . \tag{1.3.5}$$

This shows that Newton's law applies in the same way to the entire macroscopic object as it did for each of the constituent atoms, provided we include only the externally applied forces, use the macroscopic object's entire mass and identify the macroscopic object's acceleration as the acceleration of the object's centre of mass — defined by (1.3.5).

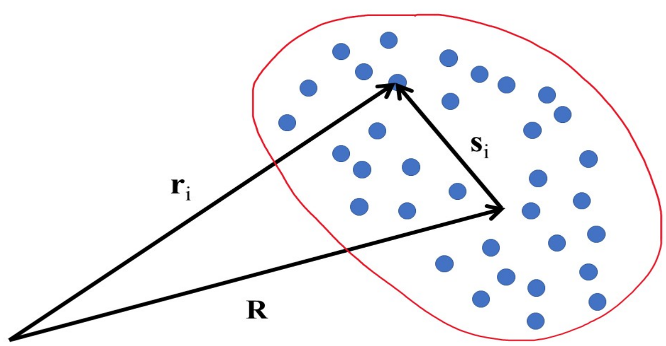


Figure 4. A sketch (not to scale) of atoms in a macroscopic object, illustrating the difference between the atomic position $\mathbf{r}_i$ and its position, $\mathbf{s}_i = \mathbf{r}_i - \mathbf{R}$, relative to the object's centre of mass, $\mathbf{R}$.

1.3.2 Recursiveness

Furthermore, this shows that Newton's 2nd law is *recursive* in the sense that it is completely agnostic about what we consider to be the underlying point particles from which the macroscopic object is made. For example suppose the object described above can be regarded as the union of two pieces, denoted A and B, so $\mathcal{O} = A \cup B$. (Maybe the macroscopic object considered above was the Earth-Moon system and A is the Earth while B is the Moon.) Then all sums over particle label a in the above argument can be broken up into sums separately over A and B:

$$M = \sum_{a=1}^{N} m_a = \sum_{a \in A} m_a + \sum_{a \in B} m_a =: M_A + M_B \,, \tag{1.3.6}$$

and similarly

$$\mathbf{F}^{\rm ext}_{\rm tot} = \sum_{a=1}^{N} \mathbf{F}^{\rm ext}_a = \sum_{a \in A} \mathbf{F}^{\rm ext}_a + \sum_{a \in B} \mathbf{F}^{\rm ext}_a =: \mathbf{F}^{\rm ext}_A + \mathbf{F}^{\rm ext}_B \,. \tag{1.3.7}$$

If we define

$$\mathbf{R}_A := \frac{1}{M_A} \sum_{a \in A} m_a \, \mathbf{r}_a \qquad \text{and} \qquad \mathbf{R}_B := \frac{1}{M_B} \sum_{a \in B} m_a \, \mathbf{r}_a \,, \tag{1.3.8}$$

then repeating the arguments leading to (1.3.3) separately for each of objects A and B implies

$$M_A \, \ddot{\mathbf{R}}_A = \mathbf{F}_{AB} + \mathbf{F}^{\rm ext}_A \qquad \text{and} \qquad M_B \, \ddot{\mathbf{R}}_B = \mathbf{F}_{BA} + \mathbf{F}^{\rm ext}_B \,, \tag{1.3.9}$$

where

$$\mathbf{F}_{AB} := \sum_{a \in A} \sum_{b \in B} \mathbf{F}_{ab} \quad \text{and} \quad \mathbf{F}_{BA} := \sum_{a \in B} \sum_{b \in A} \mathbf{F}_{ab} \tag{1.3.10}$$

describe the net forces acting between the atoms in the two larger groupings. Clearly Newton's third law $\mathbf{F}_{ab} = -\mathbf{F}_{ba}$ at the atomic level implies the same is true for the macroscopic inter-body forces:

$$\mathbf{F}_{AB} = -\mathbf{F}_{BA} \,. \tag{1.3.11}$$

With these definitions we can also ask how the centre of mass for the entire system is constructed in terms of $\mathbf{R}_A$ and $\mathbf{R}_B$. The result is very familiar:

$$M \, \ddot{\mathbf{R}} = \sum_{a=1}^{N} m_a \, \ddot{\mathbf{r}}_a = \sum_{a \in A} m_a \, \ddot{\mathbf{r}}_a + \sum_{a \in B} m_a \, \ddot{\mathbf{r}}_a = M_A \, \ddot{\mathbf{R}}_A + M_B \, \ddot{\mathbf{R}}_B \,, \tag{1.3.12}$$

where the last equality uses (1.3.8). Eqs. (1.3.12) and (1.3.9) then show that these macroscopic equations are consistent with (1.3.3).

Taken together, these arguments show that the relationship between Newton's law for the whole system and Newton's law for its two subsystems is identical to the relationship derived earlier with Newton's laws for the N atoms, specialized to the case $N = 2$. That is, it is conceptually as if each of A and B were themselves to be considered to be 'atoms'.

Clearly the same conclusion would have gone through equally well if we instead had had more than two macroscopic groupings. The result can be very useful inasmuch as it can sometimes be useful to regard macroscopic objects as themselves being point particles, such as when computing the orbital motion of planets moving around the Sun. This can be done whenever their internal structure plays no role and all that is of interest is the motion of the overall centre of mass for each of the subgroupings of particles.

This recursive nature of Newton's laws shows that the laws themselves cannot tell what the fundamental smallest objects are, since they apply equally well at *all* levels of substructure. If tomorrow evidence were to emerge that all of our atoms in eq. (1.3.1) turn out to contain still-smaller teeny-weeny proto-atoms, each of which themselves satisfy Newtons second and third laws, then nothing in the above arguments need change at all (provided we assume the position $\mathbf{r}_a$ to be an appropriately defined centre-of-mass coordinate for the atoms).

1.4 Coarse-grained Conservation Laws

The ability to group atoms together into macroscopic groups described above also goes through for conserved quantities, provided these are additive (in the sense that they are defined as a sum over contribution of each particle).

1.4.1 Linear momentum

Additivity is most obvious for linear momentum since it is so intimately tied to the motion of the centre of mass coordinate:

$$\mathbf{P} = M\dot{\mathbf{R}} = \sum_a m_a \dot{\mathbf{r}}_a = \sum_{a\in A} m_a \dot{\mathbf{r}}_a + \sum_{b\in B} m_b \dot{\mathbf{r}}_b = M_A \dot{\mathbf{R}}_A + M_B \dot{\mathbf{R}}_B = \mathbf{P}_A + \mathbf{P}_B \tag{1.4.1}$$

which shows how the definition of momentum is also recursive: because it is additive it can be regarded as the sum of the momenta of each atom or the sum of the momenta for macroscopic subsystems. Either way the result is the same as the momentum associated with the centre-of-mass motion.

Newton's laws (1.3.1) imply (1.3.3) and this tells us what the obstructions are to momentum being conserved:

$$\dot{\mathbf{P}} = \mathbf{F}^{\rm ext}_{\rm tot} \,. \tag{1.4.2}$$

$\mathbf{P}$ is conserved for any system that does not experience a net outside force: $\mathbf{F}^{\rm ext}_{\rm tot} = 0$.

1.4.2 Energy

A similar story also goes through for the total kinetic energy of motion of the constituent atoms. In this case

$$K = \sum_a \tfrac{1}{2}\, m_a\, \dot{\mathbf{r}}_a^2 = \tfrac{1}{2}\, M\, \dot{\mathbf{R}}^2 + E_{\rm int} \quad \text{where} \quad E_{\rm int} := \sum_a \tfrac{1}{2}\, m_a\, \dot{\mathbf{s}}_a^2 \,, \tag{1.4.3}$$

and $\mathbf{s}_a := \mathbf{r}_a - \mathbf{R}$ denotes the particle's position relative to the centre of mass. No linear term in $\dot{\mathbf{s}}_a$ appears in (1.4.3) due to the easily proven identity

$$\sum_{a=1}^{N} m_a \, \mathbf{s}_a = 0 \,, \tag{1.4.4}$$

that follows directly from the definition (1.3.5) of $\mathbf{R}$.

The kinetic energy of atomic motion relative to the centre of mass can be regarded as an 'internal' energy, which at this point $E_{\rm int}$ could equally well describe the energy of an overall rigid rotation of the macroscopic body, or the kinetic energy associated with altering its shape, or the random motion of its constituent atoms for an object whose macroscopic orientation and shape do not change. (More about the dynamics of rigid rotations and deformations of a macroscopic object's shape is given in §4 and §9 below.)

But we could equally well coarse grain the expression for the kinetic energy to separately sum over the objects A and B as we did above for the centre of mass. In this case eq. (1.4.3) becomes

$$K = \sum_{a \in A} \tfrac{1}{2} \, m_a \, \dot{\mathbf{r}}_a^2 + \sum_{a \in B} \tfrac{1}{2} \, m_a \, \dot{\mathbf{r}}_a^2 = K_A + K_B \,, \tag{1.4.5}$$

where

$$K_A := \sum_{a \in A} \tfrac{1}{2} \, m_a \, \dot{\mathbf{r}}_a^2 = \tfrac{1}{2} \, M_A \, \dot{\mathbf{R}}_A^2 + E_{A\,{\rm int}} \quad \hbox{where} \quad E_{A\,{\rm int}} := \sum_{a \in A} \tfrac{1}{2} \, m_a \, \dot{\mathbf{s}}_{aA}^2 \,, \tag{1.4.6}$$

and $\mathbf{s}_{aA} := \mathbf{r}_a - \mathbf{R}_A$ is the displacement from the centre of mass $\mathbf{R}_A$ of object A. A similar expression holds for K_B in terms of the deviations from its centre-of-mass position: $\mathbf{s}_{aB} = \mathbf{r}_a - \mathbf{R}_B$. Additivity of the kinetic energy again implies it is recursive: it can equally well be given as a sum over the kinetic energy of each atom, or the kinetic energy of macroscopic subsystems containing many atoms (provided these keep the internal energy of motion relative to the centre of mass).

The obstruction to energy conservation is found by using (1.3.1) to compute the rate of change of kinetic energy:

$$\begin{aligned} \dot K = \sum_a m_a \dot{\mathbf{r}}_a \cdot \ddot{\mathbf{r}}_a &= \sum_{ab} \dot{\mathbf{r}}_a \cdot \mathbf{F}_{ab} + \sum_a \dot{\mathbf{r}}_a \cdot \mathbf{F}_a^{\rm ext} \\ &= \sum_{a>b} (\dot{\mathbf{r}}_a - \dot{\mathbf{r}}_b) \cdot \mathbf{F}_{ab} + \sum_a \dot{\mathbf{r}}_a \cdot \mathbf{F}_a^{\rm ext} \,, \end{aligned} \tag{1.4.7}$$

where the last equality uses Newton's third law: $\mathbf{F}_{ab} = -\mathbf{F}_{ba}$ (note the restricted sum where $a > b$). As expected the rate of change of kinetic energy is given by the rate with which the applied forces do work on the atoms in the system. Although Newton's third law makes internal forces cancel out in the momentum evolution equation (1.4.2), it does not do the same for the work done by these forces. In general energy can be transferred to and from the relative motion of the constituent degrees of freedom.

The right-hand side of (1.4.7) has the form $\sum_a \mathbf{f}_a \cdot \dot{\mathbf{r}}_a$ where $\mathbf{f}_a := \mathbf{F}_a^{\rm ext} + \sum_b \mathbf{F}_{ab}$, but it is a famous result in vector calculus (for a quick summary see Appendix §A.4) that not all sums of differentials of the type $\sum_a \mathbf{f}_a \cdot \dot{\mathbf{r}}_a$ can be written as the time derivative of something. In order for $\sum_a \mathbf{f}_a \cdot {\rm d}\mathbf{r}_a$ to be ${\rm d}V$ for some quantity V the vectors $\mathbf{f}_a$ must have the form $\mathbf{f}_a = \nabla_a V = \partial V/\partial \mathbf{r}_a$. But if this were the case then $\nabla_a \times \mathbf{f}_a = 0$ (for each a, with no implied sum on a) so if the forces appearing in (1.4.7) were such that the $\mathbf{f}_a$'s have nonzero curl then this is an obstruction to finding a conserved energy for the system.

There are two attitudes to take about restricting attention to conservative forces. On one hand at a microscopic level everything we know about physics at the atomic level tells us that energy *is* conserved by the forces relevant at the atomic level. So from that point of view there is no loss of generality restricting attention to conservative forces.[1]

The other attitude is that life is full of friction and so one should learn to deal with it if one wishes to be a fully qualified classical mechanic. There is nothing inconsistent with physics being conservative at an atomic level and the existence of nonconservative forces at a macroscopic level. From a microscopic perspective dissipative forces like friction have their roots in the work done by interatomic forces since this allows energy to be transferred from macroscopic motions (like the motion of the centre of mass) into incoherent relative motion of particles (which can manifest macroscopically as heat when the incoherent motion is sufficiently random).[2]

1.4.3 Angular momentum

The angular momentum about a fixed origin O is also recursive in the way described above because

$$\mathbf{J} := \sum_a \mathbf{r}_a \times \mathbf{p}_a = \sum_{a\in A} \mathbf{r}_a \times \mathbf{p}_a + \sum_{a\in B} \mathbf{r}_a \times \mathbf{p}_a = \mathbf{J}_{\scriptscriptstyle A} + \mathbf{J}_{\scriptscriptstyle B}\,, \tag{1.4.8}$$

where $\mathbf{p}_a := m_a \dot{\mathbf{r}}_a$. There is in general no guarantee that the angular momentum for a macroscopic body need be expressable in terms of its linear momentum: *i.e.* $\mathbf{J}_{\scriptscriptstyle A}$ might well not be writable as $\mathbf{r} \times \mathbf{P}_{\scriptscriptstyle A}$ for some choice of lever arm $\mathbf{r}$.

The obstruction to angular momentum conservation is also found by taking the cross product of $\mathbf{r}_a$ with $m_a \ddot{\mathbf{r}}_a$ for each of eqs. (1.3.1) and summing the result, leading to

$$\begin{aligned} \dot{\mathbf{J}} = \sum \mathbf{r}_a \times m_a \ddot{\mathbf{r}}_a &= \sum_{ab} \mathbf{r}_a \times \mathbf{F}_{ab} + \sum_a \mathbf{r}_a \times \mathbf{F}_a^{\rm ext} \\ &= \sum_{a>b} (\mathbf{r}_a - \mathbf{r}_b) \times \mathbf{F}_{ab} + \sum_a \mathbf{r}_a \times \mathbf{F}_a^{\rm ext}\,, \end{aligned} \tag{1.4.9}$$

[1] We see in later sections – *e.g.* §2.6 – how to include energy-conserving forces not associated with a scalar potential (such as magnetic forces).

[2] As we see in §1.5 this description of friction is most appropriate for *kinetic* friction, that resists the relative motion of objects that slide relative to one another. *Static* friction by contrast arises from short-range interatomic forces $\mathbf{F}_{\scriptscriptstyle AB}$, acting at the interface of objects that come into contact.

where the second line uses Newton's third law to group the terms involving $\mathbf{F}_{ab}$ together with those involving $\mathbf{F}_{ba}$ (leading to the restricted sum where a must be larger than b. The right-hand side of this expression makes it convenient to define the total *internal torque* and *external torque* by

$$\boldsymbol{\tau}^{\rm int} := \sum_{ab} \mathbf{r}_a \times \mathbf{F}_{ab} \quad \text{and} \quad \boldsymbol{\tau}^{\rm ext} := \sum_a \mathbf{r}_a \times \mathbf{F}_a^{\rm ext} \,, \tag{1.4.10}$$

since these are what obstruct having $\mathbf{J}$ be conserved.

Notice that the external torque can be written more simply in the special case that the external force is proportional to a constant field. For instance for an applied constant gravitational field we have $\mathbf{F}_a^{\rm ext} = m_a \mathbf{g}$ and so $\mathbf{F}_{\rm tot}^{\rm ext} = M\,\mathbf{g}$ where $M = \sum_a m_a$ is the total mass, while

$$\boldsymbol{\tau}^{\rm ext} = \sum_a \mathbf{r}_a \times (m_a \mathbf{g}) = \left(\sum_a m_a \mathbf{r}_a \right) \times \mathbf{g} = \mathbf{R} \times (M\mathbf{g}) \tag{1.4.11}$$

where $\mathbf{R}$ is the centre-of-mass position encountered earlier in (1.3.5). This shows how a constant gravitational force exerts a torque that is equivalent to the torque that would have been experienced if the total gravitational force were applied at the centre of mass.

For a constant electric field $\mathbf{E}$ the total force exterted would instead be

$$\mathbf{F}_{\rm tot}^{\rm ext} = \sum_a q_a \, \mathbf{E} = Q\,\mathbf{E} \tag{1.4.12}$$

where q_a is the electric charge of atom 'a' and $Q = \sum_a q_a$ is the total electric charge. The net torque exerted by such an electric force for constant $\mathbf{E}$ is then

$$\boldsymbol{\tau}^{\rm ext} = \sum_a \mathbf{r}_a \times (q_a \mathbf{E}) = \left(\sum_a q_a \mathbf{r}_a \right) \times \mathbf{E} = \mathbf{R}_Q \times (Q\mathbf{E}) \,, \tag{1.4.13}$$

where the 'centre-of-charge' position, $\mathbf{R}_Q$, is defined by

$$\mathbf{R}_Q := \frac{1}{Q} \sum_a q_a \mathbf{r}_a \,. \tag{1.4.14}$$

There is no reason *a priori* why the torque $\boldsymbol{\tau}^{\rm int}$ due to internal forces must vanish, though it often does in practice. As can be seen from (1.4.9), an important special case where it does vanish is when $\mathbf{F}_{ab}$ is parallel to $\mathbf{r}_a - \mathbf{r}_b$, such as is always true if this is a conservative central force. But it can also vanish more generally for isolated systems should the physics responsible for the internal forces be rotationally invariant (more about which in §2.3).

1.5 Contact Interactions: Sliding and Rolling

The above discussion shows how Newton's Laws can be derived for macroscopic objects even if they are initially assumed only to apply to point-like constituents like atoms. This is useful because the vast majority of applications of Newton's Laws work with forces and the motion of macroscopic bodies.

Contact interactions between macroscopic objects play an important role in elementary treatments of mechanics, and these have not yet be discussed from the more microscopic perspective. To this end let us specialize to the case where two macroscopic rigid bodies, A and B, interact with each other but the forces $\mathbf{F}_{AB}$ and $\mathbf{F}_{BA} = -\mathbf{F}_{AB}$ only act over very short range compared to all of the length scales of practical interest, so the forces are irrelevant unless the bodies come into physical contact. In that case the sum over atoms in expressions like (1.3.10) only runs over atoms at the surface, S, of contact between the two bodies:

$$\mathbf{F}_{AB} = \sum_{a,b\in S} \mathbf{F}_{ab}\,. \tag{1.5.1}$$

In general this contact force can be written $\mathbf{F}_{AB} = \mathbf{N} + \mathbf{F}_{\rm sf}$ where $\mathbf{N}$ is normal to the surface of contact and $\mathbf{F}_{\rm sf}$ is defined to be parallel to this surface. ($\mathbf{F}_{\rm sf}$ is called the force of 'static friction' at the interface.) In general these forces are difficult to calculate from first principles and so are simply determined by demanding Newton's laws be consistent with the geometrical constraints and the motion that is specified at the points of contact (eg. rolling, sliding, no relative motion, and so on).

If the two objects slide along one another then $\mathbf{F}_{AB}$ is purely in the normal direction (and so $\mathbf{F}_{\rm sf} = 0$). In this case dissipative forces like dynamical friction arise as energy of motion is transferred into internal degrees of freedom (such as by generating internal heat).

On the other hand if two surfaces roll without slipping relative to one another then the surfaces of the two bodies at their points of contact do not move relative to one another. In this case the contact force need not be normal to the surface and friction arises to the extent that these forces exert a torque that acts to slow down the rolling. Rolling without slipping in this way imposes a constraint on the motion that relates how quickly the rolling object rotates to how quickly the rolling object's centre of mass moves.

For example, a cylinder of radius R rolling without slipping along a planar surface must turn through an angle $\theta = s/R$ radians if its centre of mass rolls a distance s and so the speed, $v = \dot s$, of the centre of mass (in the rest frame of the planar surface) must be related to the cylinder's angular speed of rotation, $\omega = \dot\theta$, by a constraint that is linear in these two speeds:

$$v = \omega R\,. \tag{1.5.2}$$

We return to the significance of these types of constraints in §2.5 once we first develop a few more useful tools.

1.6 Spacetime Symmetries

Symmetries turn out to play an important role in later sections, such as when identifying precisely when additive conservation laws exist. This section spells out several of the symmetries that play a role in this way (and some that do not).

In particular our focus here is on the freedom we have to redefine our origin of coordinates, O, and the orientation of the basis vectors, $\mathbf{e}_i$, (see Fig. 1) without changing the component form of Newton's laws (1.1.5):

$$m_a \begin{pmatrix} \ddot{x}_a \\ \ddot{y}_a \\ \ddot{z}_a \end{pmatrix} = \begin{pmatrix} F_{xa} \\ F_{ya} \\ F_{za} \end{pmatrix} , \tag{1.6.1}$$

where $\ddot{\mathbf{r}}_a = \ddot{x}_a \, \mathbf{e}_x + \ddot{y}_a \, \mathbf{e}_y + \ddot{z}_a \, \mathbf{e}_z$ and $\mathbf{F}_a = F_{xa} \, \mathbf{e}_x + F_{ya} \, \mathbf{e}_y + F_{za} \, \mathbf{e}_z$.

It is clear that the component forms for Newton's equations in general *do* change when performing a change of origin and/or rotation of basis vectors. For instance the equations of motion (1.2.19) found above when changing basis from $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ did *not* preserve the form (1.6.1) because (1.2.19) does not have the form

$$m_a \begin{pmatrix} \ddot{r}_a \\ \ddot{\theta}_a \\ \ddot{\phi}_a \end{pmatrix} = \begin{pmatrix} F_{ra} \\ F_{\theta a} \\ F_{\phi a} \end{pmatrix} . \tag{1.6.2}$$

The form of the component equations changed in this case because it was important to keep track of how the basis vectors themselves also changed with time.

Our focus here is on the transformations of reference frame that do not change the component form of Newton's laws (1.6.1). These form an important class of redefinitions, of which there are three different types: *translations*, *rotations* and *Galilean boosts*. The rest of this section describes each of these in turn.

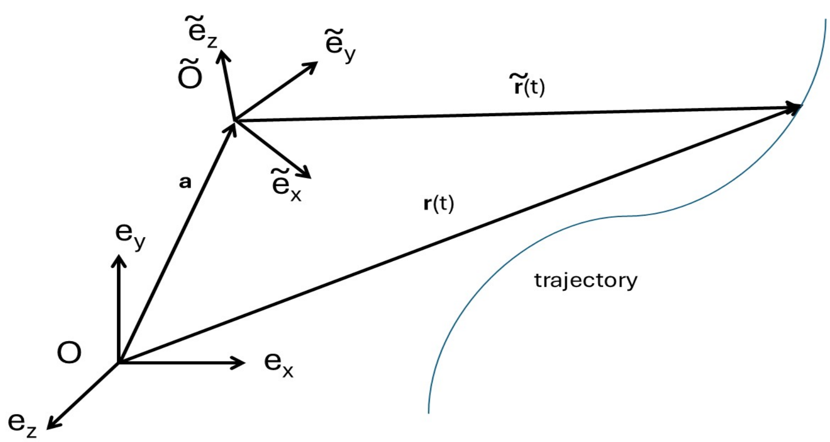


Figure 5. A sketch of the vectors $\mathbf{r}(t)$ and $\tilde{\mathbf{r}}(t)$ describing the same trajectory of a point particle relative to two different origins of coordinates, O and $\widetilde{O}$, that differ by a relative translation $\mathbf{a}$ and a rotation of basis of unit vectors from $\mathbf{e}_i$ to $\tilde{\mathbf{e}}_i$.

Translations

Consider first how Newton's law (1.6.1) changes when we shift the origin of coordinates by a constant distance $\mathbf{a}$. To this end suppose there is a second origin of coordinates $\widetilde{O}$ that is displaced from O by a displacement $\mathbf{a}$. If the position of a point of a trajectory relative to O is given by the vector $\mathbf{r}(t)$ and the position of that same point relative to $\widetilde{O}$ is denoted $\tilde{\mathbf{r}}(t)$ then inspection of Fig. 5 shows that the vectors $\mathbf{r}(t)$ and $\tilde{\mathbf{r}}(t)$ are related by

$$\mathbf{r}(t) = \tilde{\mathbf{r}}(t) + \mathbf{a}\,. \tag{1.6.3}$$

If $\mathbf{a}$ is independent of time then $\ddot{\mathbf{r}} = \ddot{\tilde{\mathbf{r}}}$ and so the left-hand side of Newton's law (1.3.1) remains unchanged. It can also be true that the right-hand side remains unchanged, such as when the forces acting amongst the particles involved depend only on the velocities of these particles and/or on their *relative* positions, $\mathbf{r}_a - \mathbf{r}_b$.

Galilean Relativity

A more general class of transformations relating O and $\widetilde{O}$ that also preserve the component form (1.6.1) of Newton's laws starts from the observation that translations need not be completely time-independent if all they must do is leave the acceleration $\ddot{\mathbf{r}}$ unchanged. Translations can also leave the acceleration unchanged if they are linear in time:

$$\mathbf{r}(t) = \tilde{\mathbf{r}}(t) + \mathbf{a} + \mathbf{u}\,t\,, \tag{1.6.4}$$

where both $\mathbf{a}$ and $\mathbf{u}$ are time-independent vectors. This is just a translation when $\mathbf{u} = 0$ but more generally describes the situation when O and $\widetilde{O}$ move relative to one another with constant relative velocity $\mathbf{u}$. In the special case $\mathbf{a} = 0$ this transformation is called a *Galilean boost*.

The right-hand side of Newton's law (1.3.1) can also be invariant under these transformations. This will automatically be true if the forces involved depend on position in a translation-invariant way. That is, translation invariance of the form (1.6.3) already requires $\mathbf{F}_{ab}$ to depend on $\mathbf{r}_a$ and $\mathbf{r}_b$ only through their difference $\mathbf{r}_a - \mathbf{r}_b$ and this is automatically also invariant under (1.6.4). But boost invariance also requires any dependence of forces on velocities $\dot{\mathbf{r}}_a$ to only depend on relative velocities, like $\dot{\mathbf{r}}_a - \dot{\mathbf{r}}_b$.

Rotations

When writing down Newton's law (1.6.1) we did not give much thought about the precise orientation of the basis vectors $\mathbf{e}_i$ we used to decompose physical quantities like positions, velocities, accelerations and forces. Yet we can ask, would the final result (1.6.1) obtained from (1.1.5) differ if a different orientation were used?

To explore this consider a new basis, $\{\tilde{\mathbf{e}}_x, \tilde{\mathbf{e}}_y, \tilde{\mathbf{e}}_z\}$, that is obtained from the old one, $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, by a rotation:

$$\tilde{\mathbf{e}}_j = \sum_{k=1}^{3} R_{jk}\,\mathbf{e}_k\,. \tag{1.6.5}$$

The requirement that both the old and new bases be orthonormal implies the coefficients R_{jk} must satisfy

$$\delta_{ij} = \tilde{\mathbf{e}}_i \cdot \tilde{\mathbf{e}}_j = \sum_{k,l=1}^{3} R_{ik}R_{jl}\,\mathbf{e}_k \cdot \mathbf{e}_l = \sum_{k=1}^{3} R_{ik}R_{jk}\,, \tag{1.6.6}$$

which can be expressed as the matrix equation

$$RR^T = R^T R = I \quad \text{or, equivalently} \quad R^{-1} = R^T\,, \tag{1.6.7}$$

where R is the matrix with elements R_{ij} (*i.e.* i is the row label and j is the column label) and R^T is its transpose with elements R_{ji} (*i.e.* j is the row label and i is the column label). I is similarly the unit matrix, whose elements are δ_{ij}. Condition (1.6.7) states that R must be an orthogonal matrix.

Since both $\tilde{\mathbf{e}}_i$ and $\mathbf{e}_i$ form a basis any vector can be written as a unique linear combination of either of them:

$$\mathbf{V} = V_x\,\mathbf{e}_x + V_y\,\mathbf{e}_y + V_z\,\mathbf{e}_z = \widetilde{V}_x\,\tilde{\mathbf{e}}_x + \widetilde{V}_y\,\tilde{\mathbf{e}}_y + \widetilde{V}_z\,\tilde{\mathbf{e}}_z\,. \tag{1.6.8}$$

Using (1.6.5) and the uniqueness of the expansion in any particular basis shows that the two component representations of $\mathbf{V}$ are related by

$$V_i = \sum_k R_{ki}\widetilde{V}_k\,. \tag{1.6.9}$$

This can be written in terms of matrix multiplication using the matrix R, with

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = R^T \begin{pmatrix} \widetilde{V}_x \\ \widetilde{V}_y \\ \widetilde{V}_z \end{pmatrix} \quad \text{or, equivalently} \quad \begin{pmatrix} \widetilde{V}_x \\ \widetilde{V}_y \\ \widetilde{V}_z \end{pmatrix} = R \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}\,. \tag{1.6.10}$$

The key question now asks whether an equation like (1.1.5) takes an identical form when decomposed in terms of either $\mathbf{e}_i$ or $\tilde{\mathbf{e}}_j$. Given the above properties of the matrix R it is simple to see that it is, *provided* the matrix R is independent of t. That is if we know that the components of position vectors and forces in two frames are related to one another by

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \widetilde{F}_x \\ \widetilde{F}_y \\ \widetilde{F}_z \end{pmatrix} = R \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}\,, \tag{1.6.11}$$

then the same relation also holds for the components of velocity and acceleration because of the assumption that R_{ij} be time-independent. If in the frame $\widetilde{O}$ the component version of (1.1.5) implies these components satisfy

$$m \begin{pmatrix} \ddot{\tilde{x}} \\ \ddot{\tilde{y}} \\ \ddot{\tilde{z}} \end{pmatrix} = \begin{pmatrix} \widetilde{F}_x \\ \widetilde{F}_y \\ \widetilde{F}_z \end{pmatrix} , \tag{1.6.12}$$

then it follows that the components in the frame O must satisfy

$$mR \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = R \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} . \tag{1.6.13}$$

But multiplication through on the left by $R^{-1} = R^T$ then implies

$$m \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} . \tag{1.6.14}$$

Comparing (1.6.12) with (1.6.14) shows that the components in these two frames satisfy precisely the same equation.

This might seem to be a fairly pedantic way to show what seems to be a fairly obvious observation about the equivalence of various components of a vector equation when compared using reference frames that differ by a rotation. But this gets the direction of implication precisely backwards: it is because the laws of physics are equivalent in rotated frames that we write the laws of physics using vectors, not the other way around. Laws of physics always have the form vector = vector or scalar = scalar (or tensor = tensor) but never have the form *e.g.* vector = scalar. And this is true precisely because it expresses that the laws of physics are invariant under rotations of the frame of reference.

In §4 we shall find other, more useful, consequences of the invariance of the laws of physics under frame rotations. In that section we instead regard the rotation as acting on the trajectory itself rather than on the reference frame, and because rotations are a symmetry of the laws of physics we will find that rotating the trajectory costs no energy. It is through arguments like this that we will come to recognize why rigid bodies are characterized by quantites like spin in addition to their centre-of-mass position.

2 Lagrangian Mechanics

This section describes the most important reformulation of classical mechanics: Lagrangian Mechanics. This new framework turns out also to be intimately related to a very beautiful and profound reformulation of the foundations of classical mechanics itself: the Principle of Least Action.

2.1 Least-Action Principle

A fundamental starting point is the reformulation of classical mechanics in more geometrical terms. Doing so naturally leads to a framework that is equally valid in arbitrary coordinates.

The Principle of Least action starts with a conservative system – *i.e.* one for which the forces arise as gradients of a potential energy, $F = -\nabla V$ – and asserts that the correct classical trajectory is the one that extremizes the value of the *action*, defined as the time integral of the difference between kinetic and potential energy:

$$S := \int \mathrm{d}t\, \Big(K - V\Big) = \int \mathrm{d}t\, L\,. \tag{2.1.1}$$

The action is so important that the integrand appearing here has its own name: $L := K - V$ is called the system's *Lagrangian*.

2.1.1 Cartesian example

To see how this works consider the Newtonian equations of motion (1.1.5) for a single point particle in the presence of a conservative force, $\mathbf{F} = -\nabla V$, defined in terms of the gradient of a scalar potential $V(\mathbf{r})$:

$$m\ddot{\mathbf{r}} = -\nabla V\,. \tag{2.1.2}$$

As we saw in §1 the kinetic energy for this system is

$$K = \tfrac{1}{2} m\dot{\mathbf{r}}^2\,. \tag{2.1.3}$$

The fundamental claim is that equations (2.1.2) are equivalent to those obtained by asking for the trajectory that extremizes the time integral of the difference between its kinetic and potential energy:

$$S[\mathbf{r}(\tau)] = \int_{t_0}^{t_f} \mathrm{d}\tau\, \left[\frac{m}{2}\,\dot{\mathbf{r}}^2 - V(\mathbf{r})\right] \tag{2.1.4}$$

subject to boundary conditions where both the initial and final positions are fixed:

$$\mathbf{r}(t_0) = \mathbf{r}_0 \quad \text{and} \quad \mathbf{r}(t_f) = \mathbf{r}_f\,. \tag{2.1.5}$$

The action (2.1.30) in this formulation should be regarded not as a function of $\mathbf{r}$ and t. Rather, it should be regarded as a *functional* of the class of paths $\mathbf{r}(t)$ that run from $\mathbf{r}_0$ to $\mathbf{r}_f$ in the time interval $t_0 \le t \le t_f$. For each such path $\mathbf{r}(t)$ eq. (2.1.30) can be evaluated and returns a real number. Different paths give different numbers and so the action is a map from the space of paths to the real numbers. Notice that nothing in the previous paragraph says that the path $\mathbf{r}(t)$ has to solve any equations of motion (like for instance (2.1.2)).

The claim to be proven is that the specific path that satisfies (2.1.2) is an extremum (*i.e.* a minimum or maximum or saddle point) of the functional $S[\mathbf{r}(t)]$. More precisely, suppose one compares the value of S when evaluated at two different paths, $\mathbf{r}(t)$ and $\tilde{\mathbf{r}}(t) = \mathbf{r}(t) + \delta\mathbf{r}(t)$, that are 'nearby' one another in the sense that $\delta\mathbf{r}(t)$ is everywhere small. The path $\mathbf{r}_c(t)$ that extremizes S is defined to be the one for which a Taylor expansion of $S[\mathbf{r}(t)+\delta\mathbf{r}(t)]$ in powers of $\delta\mathbf{r}(t)$ does not have a term linear in $\delta\mathbf{r}(t)$.

This definition of an extremal path is meant to mimic the criterion used when identifying the minimum, maximum or saddle point of an ordinary function. A point x_0 is said to extremize (*i.e.* be a minimum, maximum or saddle point of) an ordinary function $f(x)$ when the derivative $f'(x_0) = 0$ vanishes when evaluated at x_0. Equivalently, the Taylor expansion of

$$f[x + \delta x] - f(x) = f'(x)\,\delta x + \tfrac{1}{2} f''(x)\,(\delta x)^2 + \cdots \tag{2.1.6}$$

should start off at quadratic order in δx if $x = x_0$ is an extremum. The linear term in the expansion of a functional about a given path can be thought of as defining what it means to differentiate a functional, the mathematics of which is called the *Calculus of Variations* (see Appendix §A.1 for a whirlwind summary).

Notice that extremizing the action is a geometric criterion that specifies the *path* itself rather than its parameterization in terms of a specific set of coordinates. This type of formulation lends itself to a formulation of Newton's equations of motion that are equally valid in any coordinate system. And changing coordinates in a scalar function like K or V is much easier than doing so directly with vector equations of motion like (2.1.2).

To see why the equations of motion can be obtained in this way, explicitly perform the variation by evaluating $S[\mathbf{r}(\tau)]$ at a pair of nearby trajectories, $\mathbf{r}(\tau)$ and $\mathbf{r}(\tau) + \delta\mathbf{r}(\tau)$ that share their initial and final values, and so $\delta\mathbf{r}(t_0) = \delta\mathbf{r}(t_f) = 0$. The stationary configuration is the one for which $\delta S := S[\mathbf{r} + \delta\mathbf{r}] - S[\mathbf{r}]$ vanishes for arbitrary choice of $\delta\mathbf{r}(\tau)$. The action $S[\mathbf{r}(t)+\delta\mathbf{r}(t)]$ can be Taylor expanded in powers of $\delta\mathbf{r}(t)$ and the curve $\mathbf{r}(t)$ is said to extremize $S[\mathbf{r}(t)]$ if this expansion has no term linear in $\delta\mathbf{r}(t)$.

To see what it means for $S[\mathbf{r}(\tau)]$ to be stationary under small variations, expand the integrand in powers of $\delta\mathbf{r}$ and stop at linear order. Stationarity is equivalent to requiring the term linear in $\delta\mathbf{r}$ must vanish for arbitrary choices of $\delta\mathbf{r}(\tau)$, which requires:

$$\begin{aligned} 0 = \delta S &:= \int_{t_0}^{t_f} \mathrm{d}\tau \; \left\{ \left[\frac{m}{2} (\dot{\mathbf{r}} + \delta\dot{\mathbf{r}}) \cdot (\dot{\mathbf{r}} + \delta\dot{\mathbf{r}}) - V(\mathbf{r} + \delta\mathbf{r}) \right] - \left[\frac{m}{2}\, \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - V(\mathbf{r}) \right] \right\}_{\text{linear in } \delta\mathbf{r}} \\ &= \int_{t_0}^{t_f} \mathrm{d}\tau \; \Big[m\dot{\mathbf{r}} \cdot \delta\dot{\mathbf{r}} - \delta\mathbf{r} \cdot \nabla V \Big] = \Big[m\dot{\mathbf{r}} \cdot \delta\mathbf{r} \Big]_{t_0}^{t_f} - \int_{t_0}^{t_f} \mathrm{d}\tau \; \Big[m\ddot{\mathbf{r}} + \nabla V \Big] \cdot \delta\mathbf{r} \,, \end{aligned} \tag{2.1.7}$$

where the last line performs an integration by parts to trade $\delta\dot{\mathbf{r}}$ for $\delta\mathbf{r}$. We demand this to vanish for all possible $\delta\mathbf{r}(\tau)$ subject to the requirement that $\delta\mathbf{r}(t_f) = \delta\mathbf{r}(t_0) = 0$ vanishes at both endpoints. Since this boundary condition makes the surface term $\Big[m\dot{\mathbf{r}} \cdot \delta\mathbf{r} \Big]_{t_0}^{t_f}$ vanish in (2.1.7) the vanishing of the term linear in $\delta\mathbf{r}(\tau)$ requires the coefficient of $\delta\mathbf{r}$ to vanish everywhere within the integrand, which the last line of (2.1.7) shows requires[3]

$$m\ddot{\mathbf{r}} + \nabla V = 0 \quad \text{for all } t, \tag{2.1.8}$$

in agreement with the equations of motion (2.1.2).

The quantity premultiplying $\delta\mathbf{r}(\tau)$ in the last expression of (2.1.7) is called the *functional derivative* of the functional $S[\mathbf{r}(t)]$ with respect to the path $\mathbf{r}(t)$ and denoted

$$\frac{\delta S}{\delta \mathbf{r}(t)} = m\ddot{\mathbf{r}} + \nabla V\,. \tag{2.1.9}$$

Notice that the functional derivative of a functional is a function and not a number, since it is a function of t.

The variation of S can equally well be done one component at a time. Writing $K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ and $V = V(x, y, z)$ and repeating the steps taken in (2.1.7) gives

$$\begin{aligned} 0 = \delta S &:= \Big\{ S[x + \delta x, y + \delta y, z + \delta z] - S[x, y, z] \Big\}_{\text{linear in } \delta x,\, \delta y,\, \delta z} \\ &= \Big[m\Big(\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z\Big)\Big]_{t_0}^{t_f} \\ &\qquad - \int_{t_0}^{t_f} \mathrm{d}\tau \; \left[\left(m\ddot{x} + \frac{\partial V}{\partial x}\right)\delta x + \left(m\ddot{y} + \frac{\partial V}{\partial y}\right)\delta y + \left(m\ddot{z} + \frac{\partial V}{\partial z}\right)\delta z \right], \end{aligned} \tag{2.1.10}$$

and so

$$\frac{\delta S}{\delta x(t)} = m\ddot{x} + \frac{\partial V}{\partial x}\,, \quad \frac{\delta S}{\delta y(t)} = m\ddot{y} + \frac{\partial V}{\partial y} \quad \text{and} \quad \frac{\delta S}{\delta z(t)} = m\ddot{z} + \frac{\partial V}{\partial z}\,. \tag{2.1.11}$$

The condition of extremality is $\delta S/\delta x(t) = \delta S/\delta y(t) = \delta S/\delta z(t) = 0$, which again reproduces the Cartesian components of Newton's equation (1.6.1).

2.1.2 Polar coordinates

As mentioned earlier, from a purely pragmatic standpoint, rephrasing the equations of motion as a least-action principle has the advantage that it simplifies the derivation of the equations of motion in non-Cartesian coordinates.

For example, imagine changing from Cartesian coordinates $\{x^i\} = \{x, y, z\}$ to spherical polar coordinates $\{q^n\} = \{r, \theta, \phi\}$, defined by eqs. (1.2.12) (repeated here for convenience)

$$x = r\sin\theta\cos\phi\,, \quad y = r\sin\theta\sin\phi \quad \text{and} \quad z = r\cos\theta\,. \tag{2.1.12}$$

[3] Since there are many nonzero functions whose integrals are zero one might wonder why the vanishing of the integral in (2.1.7) must require that the integrand must everywhere vanish. The reason for this is δS must vanish *for all possible* $\delta\mathbf{r}$. To establish why (2.1.33) vanishes near a particular time t we simply choose $\delta\mathbf{r}(\tau)$ to vanish everywhere except in an arbitrarily small neighbourhood around $\tau = t$. This argument can be repeated for all possible choices of t (except the endpoints - more about endpoints in what follows).

Direct differentiation shows that

$$\begin{aligned}\dot{x} &= \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi \\ \dot{y} &= \dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi \qquad (2.1.13)\\ \dot{z} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta\,, \qquad (2.1.14)\end{aligned}$$

and so the kinetic energy in spherical polar coordinates is given by

$$K = \tfrac{1}{2}m\Big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\Big) = \tfrac{1}{2}m\Big(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\Big) \qquad (2.1.15)$$

The action in spherical polar coordinates for a particle with a potential energy V then is

$$S[r(t),\theta(t),\phi(t)] = \int_{t_0}^{t_f} \mathrm{d}\tau \Big[\tfrac{1}{2}m\Big(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\Big) - V\Big]\,. \qquad (2.1.16)$$

To find the conditions for extremality for this action we expand

$$\begin{aligned}\delta S &= \Big\{S[r(t)+\delta r(t),\theta(t)+\delta\theta(t),\phi(t)+\delta\phi(t)] - S[r(t),\theta(t),\phi(t)]\Big\}_{\text{linear in }\delta r,\,\delta\theta,\,\delta\phi} \\ &= \int_{t_0}^{t_f}\mathrm{d}\tau\,\Big\{m\Big[\dot{r}\,\delta\dot{r} + r^2\dot{\theta}\,\delta\dot{\theta} + r^2\sin^2\theta\,\dot{\phi}\,\delta\dot{\phi} + \Big(r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2\Big)\delta r + r^2\dot{\phi}^2\sin\theta\cos\theta\,\delta\theta\Big] \\ &\qquad - \frac{\partial V}{\partial r}\delta r - \frac{\partial V}{\partial\theta}\delta\theta - \frac{\partial V}{\partial\phi}\delta\phi\Big\} \qquad (2.1.17)\\ &= \Big[m\Big(\dot{r}\,\delta r + r^2\dot{\theta}\,\delta\theta + r^2\sin^2\theta\,\dot{\phi}\,\delta\phi\Big)\Big]_{t_0}^{t_f} + \int_{t_0}^{t_f}\mathrm{d}\tau\,\Big\{\Big[m\Big(-\ddot{r} + r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2\Big) - \frac{\partial V}{\partial r}\Big]\delta r \\ &\qquad + \Big[-\frac{\mathrm{d}}{\mathrm{d}t}\Big(mr^2\dot{\theta}\Big) + mr^2\dot{\phi}^2\sin\theta\cos\theta - \frac{\partial V}{\partial\theta}\Big]\delta\theta + \Big[-\frac{\mathrm{d}}{\mathrm{d}t}\Big(mr^2\sin^2\theta\,\dot{\phi}\Big) - \frac{\partial V}{\partial\phi}\Big]\delta\phi\Big\}\,.\end{aligned}$$

The first term vanishes because of the condition that δr, $\delta\theta$ and $\delta\phi$ all vanish at $t = t_0$ and $t = t_f$. The remaining terms vanish for all variations only if the coefficients of δr, $\delta\theta$ and $\delta\phi$ are all separately zero, leading to the following three equations:

$$0 = m\Big(-\ddot{r} + r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2\Big) - \frac{\partial V}{\partial r} \qquad (2.1.18a)$$

$$0 = -m\Big(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\Big) + mr^2\dot{\phi}^2\sin\theta\cos\theta - \frac{\partial V}{\partial\theta} \qquad (2.1.18b)$$

$$0 = -m\Big(2r\sin^2\theta\,\dot{r}\dot{\phi} + 2r^2\sin\theta\cos\theta\,\dot{\theta}\,\dot{\phi} + r^2\sin^2\theta\,\ddot{\phi}\Big) - \frac{\partial V}{\partial\phi}\,. \qquad (2.1.18c)$$

In the special case where V depends only on r these agree with eqs. (1.2.19), which were found with much more effort in §1.

2.1.3 Target-space Geometry

More generally, imagine changing variables from Cartesian coordinates $x^i(t)$ to a general set of coordinates $q^l(t)$ obtained through a general redefinition of the form $x^i(t) = f^i[q^l(t)]$ for some set of invertible functions f^i. The main restriction at this point is that the transformation is local in time: the $x^i(t)$ can be computed given only the values of the $q^l(t)$'s evaluated at the same time. The action principle formulated in this more general context has a geometrical character that helps understand why the equations of motion take the same form for arbitrary coordinate choices.

In these new coordinates the potential energy is given by $V = V(x) = V(q)$, where $V(q) = V[x(q)]$ and the potential energy is[4]

$$K = \tfrac{1}{2} m \, \delta_{ij} \, \dot{x}^i \dot{x}^i = \tfrac{1}{2} m \, \mathcal{G}_{kl}(q) \, \dot{q}^k \dot{q}^l \,, \tag{2.1.19}$$

where the quantity in the kinetic term is given explicitly by

$$\mathcal{G}_{kl}(q) = \delta_{ij} \left(\frac{\partial x^i}{\partial q^k} \right) \left(\frac{\partial x^j}{\partial q^l} \right) . \tag{2.1.20}$$

The quantity $\mathcal{G}_{kl}(q)$ is sometimes called the *target-space metric* because it is a symmetric and positive definite matrix and so can be regarded as a geometrical metric on the 'target space' (which is defined as the space in which the functions q^l take values). It plays the same role in (2.1.19) as does the metric δ_{ij} in Cartesian coordinates: $\mathcal{G}_{kl}$ quantifies an inner product between vectors and so in particular provides the 'length' of the velocity vector whose components are $\dot{q}^k$ in the same way that the Cartesian metric δ_{ij} does for the velocity vector with components $\dot{x}^i$.

This metric analogy can be made even more explicit by asking how $\mathcal{G}_{kl}(q)$ changes if one performs another coordinate change from q^k to $\tilde{q}^r = f^r(q)$, for some functions $f^r(q)$. Performing this coordinate change in (2.1.20) implies the metric in the new coordinates is related to the old one by

$$\tilde{\mathcal{G}}_{rs}(\tilde{q}) = \mathcal{G}_{kl}[q(\tilde{q})] \left(\frac{\partial q^k}{\partial \tilde{q}^r} \right) \left(\frac{\partial q^l}{\partial \tilde{q}^s} \right) , \tag{2.1.21}$$

which is recognizable as the transformation rule of a rank two covariant tensor under a general coordinate transformation. Any symmetric and positive rank two tensor can always be interpreted as a metric.

Returning to our main line of argument, the action written using the coordinates q^n is

$$S = \int_{t_0}^{t} \mathrm{d}\tau \left[\frac{m}{2} \, \mathcal{G}_{kl}(q) \, \dot{q}^k \dot{q}^l - V(q) \right] , \tag{2.1.22}$$

[4]From this point on we adopt the *Einstein summation convention* in which there is an implied sum over any repeated index. For example $\delta q^k \partial_k V$ really means $\sum_k \delta q^k \partial_k V$. This notation keeps there from being too much clutter from summation signs.

and so the condition $\delta S = 0$ reads

$$
\begin{aligned}
0 = \delta S &:= \int_{t_0}^{t} \mathrm{d}\tau \; \Big\{ \Big[\frac{m}{2} \mathcal{G}_{kl}(q+\delta q)(\dot q^k + \delta \dot q^k)(\dot q^l + \delta \dot q^l) - V(q+\delta q) \Big] \\
&\qquad\qquad - \Big[\frac{m}{2}\, \mathcal{G}_{kl}(q)\, \dot q^k \dot q^l - V(q) \Big] \Big\}_{\text{linear in } \delta q} \\
&= \int_{t_0}^{t} \mathrm{d}\tau \; \Big[m\mathcal{G}_{kl}(q)\, \dot q^l \delta \dot q^k + \frac{m}{2} \delta q^r \partial_r \mathcal{G}_{kl}(q)\, \dot q^k \dot q^l - \delta q^r \partial_r V(q) \Big] \qquad (2.1.23) \\
&= \Big[m \mathcal{G}_{kl}(q) \dot q^k \delta q^l \Big]_{t_0}^{t} + \int_{t_0}^{t} \mathrm{d}\tau \; \Big\{ -\frac{\mathrm{d}}{\mathrm{d}\tau} \Big[m \mathcal{G}_{kl}(q) \dot q^l \Big] + \frac{m}{2} \partial_k \mathcal{G}_{lr}(q) \dot q^l \dot q^r - \partial_k V(q) \Big\}\, \delta q^k \,,
\end{aligned}
$$

where ∂_r denotes $\partial / \partial q^r$.

The equations of motion are found, as usual, by asking this action to be stationary with respect to arbitrary variations that vanish at the endpoints: $\delta q^k(t_0) = \delta q^k(t_f) = 0$. The result written in terms of the coordinates q (rather than x) is

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t} \Big[m\, \mathcal{G}_{kl}(q) \dot q^l \Big] - \frac{m}{2} \partial_k \mathcal{G}_{lr}(q)\, \dot q^l \dot q^r + \partial_k V(q) \\
&= m\, \mathcal{G}_{kl}(q)\, \ddot q^l + m \Lambda_{klr}(q)\, \dot q^l \dot q^r + \partial_k V(q) \,, \qquad (2.1.24)
\end{aligned}
$$

where $\Lambda_{klr}(q)$ is the following combination of derivatives of $\mathcal{G}_{kl}(q)$

$$
\Lambda_{klr} := \tfrac{1}{2} \Big(\partial_l \mathcal{G}_{kr} + \partial_r \mathcal{G}_{kl} - \partial_k \mathcal{G}_{lr} \Big) \,. \qquad (2.1.25)
$$

This is another quantity familiar from the theory of differential geometry, called the 'Christoffel symbol of the first kind' constructed from the metric $\mathcal{G}_{kl}$.

This can be solved to give an equation for $\ddot q^k$ once the matrix $\mathcal{G}^{rs}$ that is the inverse to $\mathcal{G}_{kl}$ is found. That is, given the matrix of coefficients $\mathcal{G}^{rs}$ that satisfies $\mathcal{G}^{rk} \mathcal{G}_{kl} = \delta^r{}_l$ (with an implied sum over the repeated index k), the above equations of motion imply

$$
m \Big(\ddot q^k + \Gamma^k_{lr}(q)\, \dot q^l \dot q^r \Big) + \mathcal{G}^{kl} \partial_l V(q) = 0 \,, \qquad (2.1.26)
$$

where the 'Christoffel symbol of the second kind' is defined by

$$
\Gamma^k_{lr} := \mathcal{G}^{ks} \Lambda_{slr} := \tfrac{1}{2}\, \mathcal{G}^{ks} \Big(\partial_l \mathcal{G}_{sr} + \partial_r \mathcal{G}_{sl} - \partial_s \mathcal{G}_{lr} \Big) \,. \qquad (2.1.27)
$$

Eqs. (2.1.26) provide some insight into how the form of the equations of motion can remain the same even after performing arbitrary redefinition from q^k to $\tilde q^r = f^r(q)$. The equation found in the new variables is again of the form (2.1.26), but built using the new metric $\tilde G_{rs}$, its inverse $\tilde G^{st}$ and the corresponding Christoffel symbol $\tilde \Gamma^r_{st}$ built from these two tensors. What is important is that the metric (and its inverse) and the quantities $\partial_l V$ and $D\dot q^k = \ddot q^k + \Gamma^k_{lr}\, \dot q^l \dot q^r$ are *tensors* under coordinate transformations and so (2.1.26) is a tensor equation (in which tensors that transform in the same way are equated to one another). As a result once the equation is true for one set of coordinates it necessarily remains true in any set of coordinates, because of an argument very similar to the one made for rotations in §1.6.

2.1.4 Multiple Particles

Another straightforward generalization extends the above to include multiple particles interacting with one another and with external sources through conservative forces, so $\sum_b \mathbf{F}_{ab} = -\nabla_a V$ and $\mathbf{F}_a^{\rm ext} = -\nabla_a U$, where both U and V are functions of the particle positions $\mathbf{r}_a$, with $a = 1, \dots, N$. In this case the Newtonian equations of motion (1.3.1) become:

$$m_a \ddot{\mathbf{r}}_a = -\nabla_a \left(U + V\right) . \tag{2.1.28}$$

As we saw in §1 the kinetic energy for this system is

$$K = \tfrac{1}{2} \sum_a m_a \dot{\mathbf{r}}_a^2 \,, \tag{2.1.29}$$

and so the action in this case is

$$S[\mathbf{r}(\tau)] = \int_{t_0}^{t_f} \mathrm{d}\tau \, \left[\tfrac{1}{2} \sum_a m_a \, \dot{\mathbf{r}}_a^2 - \Big(U + V\Big) \right] \tag{2.1.30}$$

The equations of motion (1.3.1) are obtained by demanding that S be stationary subject to boundary conditions where both the initial and final positions of each particle are fixed:

$$\delta \mathbf{r}_a(t_0) = \delta \mathbf{r}_a(t_f) = 0 \,, \tag{2.1.31}$$

Proceeding as before we find

$$\begin{aligned} 0 = \delta S :=& \sum_a \int_{t_0}^{t_f} \mathrm{d}\tau \, \Big[m_a \dot{\mathbf{r}}_a \cdot \delta \dot{\mathbf{r}}_a - \delta \mathbf{r}_a \cdot \nabla_a (U+V) \Big] \\ =& \left[\sum_a m_a \dot{\mathbf{r}}_a \cdot \delta \mathbf{r}_a \right]_{t_0}^{t_f} - \sum_a \int_{t_0}^{t_f} \mathrm{d}\tau \, \Big[m_a \ddot{\mathbf{r}}_a + \nabla_a (U+V) \Big] \cdot \delta \mathbf{r}_a \,, \end{aligned} \tag{2.1.32}$$

where the surface term vanishes by virtue of (2.1.31). Requiring this to vanish for all possible choices for $\delta \mathbf{r}_a$ for all t and a then implies

$$m_a \ddot{\mathbf{r}}_a + \nabla_a (U+V) = 0 \quad \text{for all } a \text{ and all } t, \tag{2.1.33}$$

in agreement with the equations of motion (1.3.1).

2.2 Euler-Lagrange Equations

Now that we have some confidence that the least-action principle properly reproduces Newton's equations correctly it is worth identifying what these equations look like in a more general context. We therefore generalize the discussion in two separate ways:

- We allow the label 'A' of the generalized coordinate by $q^A(t)$ to run over both the components $i = 1, 2, 3$ of position in space and over the label $a = 1, \cdots, N$ that distinguishes particles from one another, so $A = \{a, i\}$.

- We allow the Lagrangian to be an arbitrary function of $q^A(t)$ and $\dot{q}^B(t)$ (evaluated at the same time), $L = L(q, \dot{q}, t)$, and do *not* assume it has the more restricted form given in (2.1.22). We also allow it to depend explicitly on time (in addition to its implicit dependence on time through the trajectories $q^A(t)$).

These more general forms allow for the possible freedom to redefine coordinates that might mix up the positions and velocities and might also mix up the coordinates for different particles.

These choices lead to the action

$$S = \int \mathrm{d}t \; L(q, \dot{q}, t) \,, \tag{2.2.1}$$

for which the variation is

$$\begin{aligned} \delta S &= \int_{t_0}^{t_f} \mathrm{d}\tau \Big\{ L[q + \delta q, \dot{q} + \delta \dot{q}] - L[q, \dot{q}] \Big\}_{\text{linear in } \delta q} \\ &= \int_{t_0}^{t_f} \mathrm{d}\tau \left[\frac{\partial L}{\partial \dot{q}^A} \, \delta \dot{q}^A + \frac{\partial L}{\partial q^A} \, \delta q^A \right] = \left[\frac{\partial L}{\partial \dot{q}^A} \, \delta q^A \right]_{t_0}^{t_f} + \int_{t_0}^{t_f} \mathrm{d}\tau \left[-\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial L}{\partial \dot{q}^A} \right) + \frac{\partial L}{\partial q^A} \right] \delta q^A \,, \end{aligned} \tag{2.2.2}$$

where there is an implied sum over the repeated index A. δS found in this way is required to vanish for arbitrary $\delta q^A(\tau)$ subject to the boundary condition

$$\delta q^A(t_0) = \delta q^A(t) = 0 \,. \tag{2.2.3}$$

The form found above for δS shows that this implies $q^A(\tau)$ satisfies the following *Euler-Lagrange equations of motion*:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial q^A} = 0 \,. \tag{2.2.4}$$

Evaluating the time derivative $\mathrm{d}/\mathrm{d}\tau$ using the chain rule then shows that the equations of motion are explicitly second-order in time derivatives:

$$\frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \ddot{q}^B + \frac{\partial^2 L}{\partial \dot{q}^A \partial q^B} \dot{q}^B + \frac{\partial^2 L}{\partial \dot{q}^A \partial t} - \frac{\partial L}{\partial q^A} = 0 \,, \tag{2.2.5}$$

where there is again an implied sum over B in the terms that involve this index. The partial time derivative $\partial/\partial t$ differentiates with both q^A and $\dot{q}^B$ held fixed (as opposed to $\mathrm{d}/\mathrm{d}\tau$, which differentiates both the explicit time-dependence in L and the implicit time-dependence hidden within $q^A(t)$). Eqs. (2.2.4) or (2.2.5) are the forms to be used throughout most of the rest of these notes.

2.3 Symmetries and Conservation Laws

One of the advantages of the least-action formulation of classical mechanics is the clear answer it provides for when to expect equations of motion to respect additive conservation laws (like conservation of energy, momentum and angular momentum): they are tied to the existence of symmetries (as this section shows explicitly).

To start things off it is useful to identify the immediate analogs of the momentum and energy in the case when we use generalized coordinates like q^A. We seek functions of $F(q, \dot{q}, t)$ that are constants when q and $\dot{q}$ are evolved using the classical Euler-Lagrange equations (2.2.4). That is, F should satisfy

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial \dot{q}^A}\,\ddot{q}^A + \frac{\partial F}{\partial q^A}\,\dot{q}^A + \frac{\partial F}{\partial t} = 0 \quad \text{when } q(t) \text{ satisfies} \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial q^A} = 0\,. \qquad (2.3.1)$$

To keep the result additive when including many particles we start by seeking choices for F that are linear in the Lagrangian since (as we shall see) it is the kinetic part of L that matters and this is often additive – see *e.g.* eq. (2.1.30).

2.3.1 Generalized Momenta

A first guess can be called the generalized momentum, p_A, defined by

$$p_A := \frac{\partial L}{\partial \dot{q}^A}\,. \qquad (2.3.2)$$

Inspection of the equation of motion (2.2.4) shows that this satisfies

$$\frac{\mathrm{d}p_A}{\mathrm{d}t} = \frac{\partial L}{\partial q^A} \qquad (2.3.3)$$

when the equations of motion hold and so p_A is conserved for any coordinate q^A that only appears in L through its derivate $\dot{q}^A$. Any generalized coordinate that only appears differentiated in L is called an *ignorable coordinate*, because – as we see in more detail in §7 below – its influence on the other degrees of freedom can be summarized by the constant value taken by the corresponding generalized momentum in the equations of motion.[5]

We have already seen two examples of this. The first of these is the two-body problem studied in §1.2.1 with a mutual force that depends only on the relative separation $\mathbf{r}_1 - \mathbf{r}_2$. In this case the coordinates are $\{q^A\} = \{\mathbf{r}_1, \mathbf{r}_2\}$ and the Lagrangian is the $N = 2$ special case of (2.1.30):

$$L = \tfrac{1}{2}\Big(m_1\dot{\mathbf{r}}_1^2 + m_2\dot{\mathbf{r}}_2^2\Big) - V(\mathbf{r}_1 - \mathbf{r}_2)\,. \qquad (2.3.4)$$

This becomes an example of conserved generalized momentum if we change variables to centre-of-mass and relative positions, $\{q^A\} = \{\mathbf{R}, \mathbf{r}\}$ (see the discussion surrounding (1.2.6)), in which case

$$L = \tfrac{1}{2}\Big(M\dot{\mathbf{R}}^2 + \mu\dot{\mathbf{r}}^2\Big) - V(\mathbf{r})\,, \qquad (2.3.5)$$

[5] It is worth noting in passing that although the effects of an ignorable coordinate on the evolution of other variables amounts to replacing their momenta with the conserved constant to which they are equal, this is only true if it is done within the equations of motion. This is *not* equivalent to making this replacement in the Lagrangian itself before deriving the equations of motion.

where M and μ are respectively the system's total and reduced masses (defined in (1.2.5) and (1.2.9)). In terms of these variables the variable $\mathbf{R}$ enters L only through $\dot{\mathbf{R}}$ and so the corresponding generalized momentum

$$\mathbf{P} = \frac{\partial L}{\partial \dot{\mathbf{R}}} = M\dot{\mathbf{R}} = m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2 \,, \tag{2.3.6}$$

is conserved (compare with (1.2.7)).

This same example also contains the second case of conserved generalized momentum, as can be seen when the relative position $\mathbf{r}(t)$ is expressed in terms of spherical polar coordinates: $\{x(t), y(t), z(t)\} \to \{r(t), \theta(t), \phi(t)\}$ using (1.2.12). In this case the $\mathbf{r}$-dependent part of L is given by (2.1.16) with $V = V(r)$. This shows that the variable $\phi(t)$ only appears in L through $\dot{\phi}$ and as a result its generalized momentum

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = r^2 \sin^2\theta \, \dot{\phi} \tag{2.3.7}$$

is conserved. Once coordinate are adapted to ensure $\theta(t) = \frac{\pi}{2}$ for all t we see that p_ϕ conservation is equivalent to conservation of the magnitude of angular momentum $J = |\mathbf{J}|$ (see (1.2.21)).

This illustrates the power of the least-action formulation: because the equations of motion (2.2.4) are equally valid for *any* choice of generalized coordinate we are free to use the freedom to redefine variables to make as many coordinates as possible ignorable, in which case their evolution comes down to the conservation of an appropriate generalized momentum.

2.3.2 Energy Conservation

A second very general guess for a conserved quantity is the energy, defined as

$$E := \frac{\partial L}{\partial \dot{q}^A} \, \dot{q}^A - L = p_A \dot{q}^A - L \,, \tag{2.3.8}$$

where the Einstein summation convention is in force so there is an implied sum over the repeated index A. To see when and why this might be conserved we compute its derivative with respect to time, using (2.2.4) to simplify the result:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^A}\right) \dot{q}^A + \frac{\partial L}{\partial \dot{q}^A} \, \ddot{q}^A - \left(\frac{\partial L}{\partial \dot{q}^A} \, \ddot{q}^A + \frac{\partial L}{\partial q^A} \, \dot{q}^A + \frac{\partial L}{\partial t}\right) = -\frac{\partial L}{\partial t} \,, \tag{2.3.9}$$

where the second equality eliminates $(\mathrm{d}/\mathrm{d}t)(\partial L/\partial \dot{q}^A)$ using (2.2.4). We see that E is very generally conserved whenever L does not depend on t apart from the implicit t-dependence associated with the path $q(t)$.

To see why (2.3.8) can be interpreted as energy, evaluate it for the special case of the two-body Lagrangian given in (2.3.4). Using coordinates $\{q^A(t)\} = \{\mathbf{r}_1(t), \mathbf{r}_2(t)\}$ we have

$$\frac{\partial L}{\partial \dot{q}^A} \, \dot{q}^A = \frac{\partial L}{\partial \dot{\mathbf{r}}_1} \cdot \dot{\mathbf{r}}_1 + \frac{\partial L}{\partial \dot{\mathbf{r}}_2} \cdot \dot{\mathbf{r}}_2 = m_1 \dot{\mathbf{r}}_1^2 + m_2 \dot{\mathbf{r}}_2^2 \tag{2.3.10}$$

and so

$$E = \left(\frac{\partial L}{\partial \dot{\mathbf{r}}_1} \cdot \dot{\mathbf{r}}_1 + \frac{\partial L}{\partial \dot{\mathbf{r}}_2} \cdot \dot{\mathbf{r}}_2 \right) - L = \tfrac{1}{2} m_2 \dot{\mathbf{r}}_1^2 + \tfrac{1}{2} \dot{\mathbf{r}}_2^2 + V \,, \tag{2.3.11}$$

which is to be compared with the $U = 0$ limit of (1.2.49).

2.3.3 Noether's Theorem

There is a very deep reason behind the existence of these conservation laws: each one is associated with a continuous symmetry. This connection between symmetries and conservation laws is called *Noether's theorem*.

For the present purposes a symmetry is a change of the path $q^A(t) \to \tilde{q}^{\,A}(t)$ that leaves the action completely unchanged:

$$S[q(t)] \equiv S[\tilde{q}(t)] \quad \text{for all } q(t) \,. \tag{2.3.12}$$

It is important in what follows that the above expression applies *for all choices of initial path*, $q(t)$. For example, the action

$$S[\mathbf{r}(t)] = \int_{t_0}^{t_f} \mathrm{d}\tau \, \left[\tfrac{1}{2} m \dot{\mathbf{r}}^2 - V(|\mathbf{r}|) \right] \tag{2.3.13}$$

encountered in earlier sections is invariant under $\mathbf{r}(t) \to -\mathbf{r}(t)$ for any initial choice of path $\mathbf{r}(t)$. (This particular symmetry under reflection of coordinates is called *parity*.)

A *continuous* symmetry is a symmetry for which there is a family of such changes, $q^A(t) \to \tilde{q}^{\,A}(\omega, t)$ for which the the action remains unchanged for all values of a continuous parameter ω. It is conventional to choose the parameter such that the particular choice $\omega = 0$ corresponds to the trivial transformation for which $\tilde{q}(0, q) = q$. It is often useful to restrict to transformations that are arbitrarily close to the trivial transformation, for which $\omega = \epsilon \ll 0$, since in this case the transformation rule can be Taylor expanded in powers of ϵ:

$$\delta_\epsilon q^A := \tilde{q}^A(\epsilon, q) - q^A = \epsilon \left(\frac{\partial \tilde{q}^A}{\partial \omega} \right)_{\omega = 0} + \mathcal{O}(\epsilon^2) \,. \tag{2.3.14}$$

There can of course be more than one parameter's worth of symmetries. §1.6 already provides several examples of continuous symmetries: the three-parameter family of spacetime symmetries of translation $\mathbf{r} \to \mathbf{r} + \mathbf{a}$; the three-parameter family of rotations $\mathbf{r} \to R\,\mathbf{r}$ with $R^T R = I$ and the three-parameter family of Galilean boosts $\mathbf{r} \to \mathbf{r} + \mathbf{u}\, t$. In these examples the continous parameters are the three components of the vectors $\mathbf{a}$ and $\mathbf{u}$ and the three angles (more about which below) that define any 3×3 orthogonal matrix R. When more than one parameter is available we gather them all into a collection ω^α, where the index α runs from 1 to the total number of independent continuous parameters. In this case (2.3.14) generalizes to

$$\delta_\epsilon q^A := \tilde{q}^A(\epsilon, q) - q^A = \epsilon^\alpha f^A_\alpha(q) + \mathcal{O}(\epsilon^2) \quad \text{where} \quad f^A_\alpha := \left(\frac{\partial \tilde{q}^A}{\partial \omega^\alpha} \right)_{\omega = 0} \,, \tag{2.3.15}$$

and there is an implied sum (as always) over the repeated indices 'α' and we specialize to the infinitesimal case $\omega^\alpha = \epsilon^\alpha$ where each of the ϵ^α's is assumed small enough to justify neglecting quadratic and higher terms.

Whenever there is a continuous symmetry the action satisfies

$$S[q] \equiv S[\tilde{q}(\omega, q)] \quad \text{for all } \omega^\alpha \text{ and for all } q^A(t). \tag{2.3.16}$$

Specializing to a transformation close to the trivial one, $\omega^\alpha = \epsilon^\alpha$, and Taylor expanding in powers of ϵ^α leads the leading term of (2.3.16) to be written

$$\begin{aligned} 0 \equiv \delta S[q] &:= S[\tilde{q}(\epsilon, q)] - S[q] = \epsilon^\alpha \int_{t_0}^{t_f} \mathrm{d}\tau \left[\left(\frac{\partial L}{\partial \dot{q}^A} \right) \dot{f}^A_\alpha + \left(\frac{\partial L}{\partial q^A} \right) f^A_\alpha \right] \\ &= \epsilon^\alpha \left[\left(\frac{\partial L}{\partial \dot{q}^A} \right) f^A_\alpha \right]_{t_0}^{t_f} + \epsilon^\alpha \int_{t_0}^{t_f} \mathrm{d}\tau\, f^A_\alpha \left[-\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial L}{\partial \dot{q}^A} \right) + \left(\frac{\partial L}{\partial q^A} \right) \right] . \end{aligned} \tag{2.3.17}$$

where we assume ϵ^α does not depend on t (since any t-dependence can be lumped into $f^A_\alpha(q)$) and integrate by parts to remove the derivative from $\dot{f}^A_\alpha$. The zero in the first equality is a consequence of the transformation (2.3.15) being a symmetry for which (2.3.16) is true. What is important is the symmetry condition (2.3.16) implies eq. (2.3.17) is an identity that holds for *all* paths $q^A(t)$ and all parameters ϵ^α.

The final step is to specialize (2.3.17) to the special case where $q^A(t) = \overline{q}^A(t)$ satisfies the Euler-Lagrange equation (2.2.4):

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial L}{\partial \dot{q}^A} \right) - \left(\frac{\partial L}{\partial q^A} \right) \right]_{\overline{q}(t)} = 0 \,. \tag{2.3.18}$$

In this case the integral in the second line of (2.3.17) vanishes and the remainder implies

$$Q_\alpha(t_f) = Q_\alpha(t_0) \quad \text{where} \quad Q_\alpha(t) := \left(\frac{\partial L}{\partial \dot{q}^A} \right) f^A_\alpha \,. \tag{2.3.19}$$

This is a conservation law – it states that Q_α remains unchanged from t_0 to t_f – and it is additive to the extent that L itself is. And it is constructive: once the Lagrangian and the symmetry transformation (2.3.15) is given eq. (2.3.19) provides an explicit formula for the conserved quantity Q_α – called the *Noether charge* – as a function of q^A and $\dot{q}^A$. The construction shows that there is one conserved charge for each independent continuous symmetry.

2.3.4 Examples

The symmetries encountered to this point can now in retrospect be identified as the consequences of symmetries.

- Conservation of generalized momentum p_A encountered in §2.3.1 when L is independent of q^A can now be seen as a consequence of L having a shift symmetry: L remains unchanged if $q^A \to q^A + \text{constant}$.

- Conservation of energy as derived in §2.3.2 can similarly be seen as a consequence of L being invariant under time translation: $t \to t + \text{constant}$, which in particular implies $\partial L/\partial t = 0$.

- Conservation of linear momentum $\mathbf{P}$ is a special case of conservation of generalized momentum when the generalized coordinate is simply the Cartesian coordinate for each particle in the system: $\{q^A\} = \{x^{ai}\}$ where x^{ai} for $i = 1, 2, 3$ are the Cartesian components of the position vectors $\mathbf{r}_a$ for each particle relative to the basis vectors $\mathbf{e}_i = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. Here the label '$a = 1, \cdots, N$' runs over the different particles present. Conservation of linear momentum therefore follows from the symmetry of translation invariance, under which all particle positions shift by the same amount: $\mathbf{r}_a \to \mathbf{r}_a + \mathbf{a}$. Equivalently, this corresponds to a constant shift $\mathbf{R} \to \mathbf{R} + \mathbf{a}$ for the centre-of-mass $\mathbf{R} = \sum_a m_a \mathbf{r}_a$ with all relative positions $\mathbf{r}_a - \mathbf{r}_b$ held fixed.

- Conservation of $p_\phi = mr^2\dot{\phi}$ for the central-force problem studied in §1.2.1 is also a special case of conservation of generalized momentum which we've seen corresponds to the conservation of the length J of the angular momentum vector $\mathbf{J} = \mathbf{r} \times \mathbf{p}$. In this case the symmetry is $\phi \to \phi + \text{constant}$, which is a rotation about the $\mathbf{e}_z$ axis (that lies perpendicular to the plane of the particle's orbit).

More generally, for a general rotation R, choosing the rotation to be very close to the trivial transformation corresponds to writing $R_{ij} = \delta_{ij} + \Theta_{ij}$ where the components of Θ_{ij} are very small. Expanding the orthogonality condition $R^T R = I$ – see (1.6.7) – out to linear order then implies the matrix with components Θ_{ij} must be antisymmetric: $\Theta_{ij} = -\Theta_{ji}$.

Any 3×3 antisymmetric matrix has only 3 independent components, which is the same as for a vector in 3 dimensions. This is no accident: the matrix Θ_{ij} can be related to a vector $\boldsymbol{\Theta}$ with components Θ_k by writing $\Theta_{ij} = \sum_k \epsilon_{ijk}\,\Theta_k$, where ϵ_{ijk} is the *Levi-Civita* tensor, which is defined to be completely antisymmetric under the interchange of any pair of indices and by convention is chosen with $\epsilon_{123} = \epsilon_{xyz} = +1$ (see Appendix A.3.1). In matrix form this reads

$$\begin{pmatrix} 0 & \Theta_{xy} & \Theta_{xz} \\ \Theta_{yx} & 0 & \Theta_{yz} \\ \Theta_{zx} & \Theta_{zy} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Theta_z & -\Theta_y \\ -\Theta_z & 0 & \Theta_x \\ \Theta_y & -\Theta_x & 0 \end{pmatrix} . \tag{2.3.20}$$

We saw in §1.6 that a rotation of the basis vectors $\tilde{\mathbf{e}}_i = \sum_j R_{ij}\mathbf{e}_j$ implies the components of a vector $\mathbf{V}$ relative to these bases are related to one another by $\widetilde{V}_i = \sum_j R_{ij}V_j$ (see eq. (1.6.10)). For an infinitesimal rotation this becomes $\delta V_i = \widetilde{V}_i - V_i = \sum_j \Theta_{ij}V_j$. In terms of the vector $\boldsymbol{\Theta}$ this becomes $\delta V_i = \sum_{jk} \epsilon_{ijk} V_j \Theta_k$, which is just the component version of the vector equation

$$\delta\mathbf{V} = \mathbf{V} \times \boldsymbol{\Theta} \qquad \text{(passive rotation)} . \tag{2.3.21}$$

Now comes the main point. Suppose the Lagrangian $L(\dot{\mathbf{r}}, \mathbf{r})$ is invariant under a rotation where – in the spirit of the Noether theorem's proof – we actively rotate the particle's path (as opposed to passively rotating the basis vectors $\mathbf{e}_i$ for fixed particle path).[6]

$$\delta\mathbf{r} = \boldsymbol{\Theta} \times \mathbf{r} \qquad \text{(active rotation)}, \tag{2.3.22}$$

corresponding to an *active* rotation of $\mathbf{r} = x_i\, \mathbf{e}_i$, where the three components Θ_i of $\boldsymbol{\Theta}$ are the three continuous symmetry parameters. Then Noether tells us to expect three conserved charges, Q_i, given by (2.3.19), which in this case takes the form

$$Q_i(t) := \left(\frac{\partial L}{\partial \dot{x}^j}\right) f_i^j = p_j\, f_i^j\,, \tag{2.3.23}$$

with the usual implied sum over j and where p_j are the components of the momentum $\mathbf{p}$. The components f_i^j can be read off from (2.3.15), which in this case reads

$$\delta_\Theta x^j = \Theta^i f_i^j = \epsilon^j{}_{ik}\Theta^i x^k \quad \text{and so} \quad f_i^j = \epsilon^j{}_{ik} x^k\,. \tag{2.3.24}$$

The conserved charge that follows from rotation invariance therefore is

$$Q_i = \epsilon^j{}_{ik} x^k p_j = \epsilon_{ikj} x^k p^j \quad \text{which is the component version of} \quad \mathbf{Q} = \mathbf{r} \times \mathbf{p} = \mathbf{J}\,. \tag{2.3.25}$$

This shows that conservation of the angular momentum vector is a consequence of rotation invariance.

2.4 Ambiguities in L

There is one symmetry discussed in §1.6 for which we have not yet encountered a conservation law: Galilean boosts. For instance consider N particles interacting through a potential V that is a function only of inter-particle displacements: $\mathbf{r}_a - \mathbf{r}_b$. This is invariant under the boosts $\delta\mathbf{r}_a = \mathbf{u}\, t$ where $\mathbf{r}_a$ is the position of the a'th particle because the boost cancels in $\mathbf{r}_a - \mathbf{r}_b$. Boosts also do not change the acceleration $\delta\ddot{\mathbf{r}}_a = 0$ and so the equations of motion $m_a\ddot{\mathbf{r}}_a + \nabla_a V = 0$ are invariant (as claimed in §1.6).

We discuss this case separately here because it turns out *not* to produce a useful conservation law. This section aims to determine why not, and to understand when this occurs more generally. It also provides an excuse for examining more explicitly how much freedom

[6] The distinction between active and passive transformations is subtle but important. In a passive rotation all physical vectors are held fixed but one rotates only the basis vectors with respect to which a vector's components are defined. An active transformation instead rotates all vectors, much as one would do if one were on a rotating object (see §3.2). An active transformation (with basis vectors held fixed) has the same effect as a passive rotation, but with the rotation in the opposite direction. If the transformation rule for the components of $\mathbf{V}$ under a passive rotation is $\widetilde{V}_i = R_{ij}V_j$ (as derived above) then the transformation rule for components under the same active rotation would be $V_i \to (R^{-1})_{ij}V_j = (R^T)_{ij}V_j = R_{ji}V_j$ (and so $\delta\mathbf{V}$ has the opposite sign).

there is beyond symmetries to change a system's Lagrangian without changing its equations of motion.

Unlike the cases considered above, Galilean boosts are examples of a symmetry of the equations of motion that are *not* symmetries of the Lagrangian. Although the potential V is invariant, boosts are *not* symmetries of the kinetic energy because $\dot{\mathbf{r}}_a \to \dot{\mathbf{r}}_a + \mathbf{u}$. The Lagrangian changes under an infinitesimal boost by

$$\delta L = \widetilde{L} - L = \sum_a \tfrac{1}{2}\, m_a \Big[(\dot{\mathbf{r}}_a + \mathbf{u})^2 - \dot{\mathbf{r}}_a^2 \Big] = \mathbf{u} \cdot \sum_a m_a \dot{\mathbf{r}}_a + \mathcal{O}(\mathbf{u}^2)\,. \tag{2.4.1}$$

How can a symmetry of the equations of motion not be a symmetry of L? What is important is that L changes by a total time derivative

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}t} \left[\mathbf{u} \cdot \sum_a m_a \mathbf{r}_a \right]\,, \tag{2.4.2}$$

and so the action changes only at the endpoints

$$\delta S = \int_{t_0}^{t_f} \mathrm{d}\tau \; \delta L = \left[\mathbf{u} \cdot \sum_a m_a \mathbf{r}_a \right]_{t_0}^{t_f}\,. \tag{2.4.3}$$

Because this does not depend on $\mathbf{r}(t)$ for $t \neq t_0,\, t_f$ it cannot change the Euler-Lagrange equations obtained by varying $\mathbf{r}$ at these intermediate times.

Not having the action be invariant means the first equality in (2.3.17) that started off the proof of Noether's theorem does not go through as initially argued. To fix this let us suppose that under the transformation $\delta q^A = \epsilon^\alpha f^A_\alpha$ the Lagrangian satisfies

$$\delta L = \epsilon^\alpha \frac{\mathrm{d}W_\alpha}{\mathrm{d}t} \quad \text{for some } W_\alpha(q, \dot{q}, t)\,, \tag{2.4.4}$$

and so $\delta S = \int_{t_0}^{t_f} \mathrm{d}\tau \, \delta L = \epsilon^\alpha [W_\alpha(t_f) - W_\alpha(t_0)]$ instead of $\delta S = 0$ as was previously assumed. Under these circumstances eq. (2.3.17) is instead replaced by

$$W_\alpha(t_f) - W_\alpha(t_0) = \left[\left(\frac{\partial L}{\partial \dot{q}^A} \right) f^A_\alpha \right]_{t_0}^{t_f} + \int_{t_0}^{t_f} \mathrm{d}\tau \, f^A_\alpha \left[-\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial L}{\partial \dot{q}^A} \right) + \left(\frac{\partial L}{\partial q^A} \right) \right]\,. \tag{2.4.5}$$

Once evaluated at a solution to the Euler-Lagrange equation (2.2.4) this still leads to a conservation law, though one that differs from (2.3.19):

$$Q_\alpha(t_f) - W_\alpha(t_f) = Q_\alpha(t_0) - W_\alpha(t_0)\,, \tag{2.4.6}$$

with Q_α again as given in (2.3.19). The total conserved charge in this case becomes

$$Q_\alpha - W_\alpha = \left(\frac{\partial L}{\partial \dot{q}^A} \right) f^A_\alpha - W_\alpha\,. \tag{2.4.7}$$

What does this mean for the case of Galilean boosts? In this case we have $\{q^A\} \to \{x^{ia}\}$ given by the Cartesian components of the position vectors $\mathbf{r}_a$ and the symmetry parameters are the three components of $\mathbf{u}$ so we have

$$W_i = \sum_a m_a x_{ia} \quad \text{and} \quad f_i^{ja} = \delta_i^j \, t \quad \text{(independent of } a\text{)} \,. \tag{2.4.8}$$

In this case the Noether charge predicted by (2.4.6) becomes $Q_i - W_i = \sum_a [p_{ia} t - m_a x_{ia}]$ and so

$$\mathbf{Q} - \mathbf{W} = \sum_a \Big(\mathbf{p}_a \, t - m_a \mathbf{r}_a \Big) = \mathbf{P} \, t - M\mathbf{R} \,, \tag{2.4.9}$$

where $M = \sum_a m_a$ is the total mass, $\mathbf{R} = M^{-1} \sum_a m_a \mathbf{r}_a$ is the centre-of-mass position and $\mathbf{P} = M\dot{\mathbf{R}}$ is the centre-of-mass momentum (as defined in §1.3.1).

We see from this that the conclusion of Noether's theorem is indeed true, since this states that

$$\mathbf{P}(t) \, t - M\mathbf{R}(t) \tag{2.4.10}$$

should be independent of t. This is indeed a true statement since our assumption that the potential energy V depends only on position differences $\mathbf{r}_a - \mathbf{r}_b$ implies the centre of mass experiences no external force, so the general solution to the centre-of-mass motion is that of a free particle: $\mathbf{R} = \mathbf{R}_0 + \mathbf{V}_0(t - t_0)$ (compare with the $\mathbf{g} = 0$ limit of (1.2.10)). Consequently $\mathbf{P} = M\dot{\mathbf{R}} = M\mathbf{V}_0$ is constant and $M\mathbf{R}(t) = M\mathbf{R}_0 + \mathbf{P}(t - t_0)$ for all t and so $\mathbf{P}\, t - M\mathbf{R}(t) = \mathbf{P}\, t_0 - M\mathbf{R}_0$ is time-independent, as claimed. This result is not that useful as a conservation law, however, because it doesn't really say much more beyond what already follows from momentum conservation. The extension of translation invariance to include time-dependent translations (such as boosts) doesn't add much new by way of conservation laws, due to the explicit dependence of the 'conserved charge' on t.

This section teaches us two things.

- Invariance of the action under a continuous symmetry always implies a conservation law, though this conservation law is more useful if the symmetry transformation $\delta q^A = \epsilon^\alpha f^A_\alpha(q, \dot{q}, t)$ does not itself depend on time: $\partial f^A_\alpha / \partial t = 0$. Because the conserved quantity is linear in $\partial L / \partial \dot{q}^A$ it is additive whenever the kinetic part of L is.[7]
- Invariance of the equations of motion and of the action under a symmetry can, but need not, require the Lagrangian to be invariant, provided the variation of the Lagrangian is a total time derivative: $\delta L = \mathrm{d}W/\mathrm{d}t$ for some $W(q, \dot{q})$. If the Lagrangian is not invariant then it is important when constructing the conserved charge to keep track of W – *i.e.* use (2.4.6) rather than (2.3.19)).

[7]This statement is cleanest when $L = K - V$ but becomes less clear cut if L is not of this form.

The second bullet point is part of a broader observation that is useful to keep in mind when trying to guess the Lagrangian for a given system: there is not a unique Lagrangian that produces any given set of Euler-Lagrange equations. In particular, two Lagrangians that differ by a total time derivative – *i.e.*

$$L_1(q, \dot{q}, t) = L_2(q, \dot{q}, t) + \frac{\mathrm{d}G}{\mathrm{d}t} \tag{2.4.11}$$

for some choice for $G(q, \dot{q}, t)$ – produce exactly the same Euler-Lagrange equations (though can give different conditions for stationarity at the endpoints if δq^A is allowed to vary at the initial and final times).

A second example of different Lagrangians giving the same equations of motion is

$$L_1(q, \dot{q}, t) = \lambda L_2(q, \dot{q}, t)\,, \tag{2.4.12}$$

where λ is some constant. This type of rescaling of L leaves the equations of motion (2.2.4) completely unchanged. This sometimes can be useful, such as if the Lagrangian is a homogeneous function of the variables in the sense that there exist two constants a and b such that if we scale $q \to \lambda q$ and $t \to \lambda^a t$ then $L \to \lambda^b L$ for arbitrary constant λ. For example if $K = \frac{1}{2} m \delta_{ij} \dot{x}^i \dot{x}^j$ then

$$K \to \lambda^{2-2a} K \quad \text{if we scale } x^i \to \lambda x^i \text{ and } t \to \lambda^a t. \tag{2.4.13}$$

If the potential scales the same way then so too does L and this means that the rescaling is a symmetry of the equations of motion.

If, for instance $V \to \lambda^2 V$, as would be appropriate if V were purely quadratic in the x^i (such as if $V \propto r^2$ where $r = \sqrt{x^2 + y^2 + z^2}$ or any other quadratic combination of positions) then V scales the same way as does K if $a = 0$. Rescaling harmonic oscillator positions but not scaling time is a symmetry of the equations and this is why the period of a harmonic oscillator does not depend on the amplitude of the oscillation.

Similarly, if V is linear in the position coordinates x^i (such as is true for the gravitational force $\mathbf{F}_g = m\mathbf{g}$, for which $V = mgz$) then $V \to \lambda V$ as $x^i \to \lambda x^i$. This scales the same way as does K if $a = \frac{1}{2}$, and so if a particle is allowed to fall from rest under the influence of a constant gravitational field the distance travelled is a quadratic function of time (as it famously is – see eq. (1.1.6)).

Alternatively, if $V \propto 1/r$ where $r = \sqrt{x^2 + y^2 + z^2}$ then $V \to \lambda^{-1} V$, which is the same scaling as for K if $a = \frac{3}{2}$. Any periodic solution to the Kepler problem of finding orbits for an inverse-square force can be rescaled to a new one provided $t \to \lambda^{3/2} t$ when $x^i \to \lambda x^i$. This is why Kepler's third law relates the period P and semimajor axis a of an orbit by $P \propto a^{3/2}$ (or $P^2 \propto a^3$, as it is usually stated).

2.5 Constraints

An important role in classical mechanics is played by contraints: relationships imposed independent of the equations of motion that relate the system variables q^A and $\dot{q}^A$. One of the virtues of the least-action principle is the relative ease with which such constraints can be incorporated into the analysis of motion.

Examples of constraints arise when examining even very simple systems. Examples include weights connected by ropes on pulleys (in which case the fixed length of the rope relates the motion of one weight to the motion of the others); a ball that rolls without slipping along a surface (in which case the constraint relates the distance the centre of mass travels to the angle through which the ball rolls – see *e.g.* eq. (1.5.2)); or a bead that is trapped to move along a wire of a fixed shape. Constraints also arise whenever the motion of a macroscopic 'rigid body' is described, in which internal forces are assumed to constrain each constituent atom not to move relative to all the others so that all that matters is the motion of the centre of mass and rotation about this centre of mass (more about this in §4).

In each of these examples we understand the constraint to be a phenomenological expression that in principle could be derived given a better understanding of all of the forces that act between all of the atoms within the system (rope, ball or bead) of interest. These forces are complicated but all that matters is that their net effect is to impose a large energy cost for allowing the atoms to move relative to one another. The potential energy as a function of the positions of all of the atoms is a very complicated function but within the landscape of its minima and maxima lies a deep and narrow trough along which the relative positions of the atoms is fixed.

This picture of the interatomic potential fits well with our intuition about inter-atomic forces: they are often weakly attractive when the atoms are widely separated (because of the ability of their internal electrons to adjust in the presence of the electric fields sourced by the electrons of other atoms) but are strongly repulsive at short distances (because the Pauli exclusion principle discourages electrons from being too near one another). What is important is that this potential does not similarly constrain things like centre-of-mass position or angular orientation, and as we see in §4 there is a good symmetry reason for why this is so.

The good news is that the motion of macroscopic bodies does not require a detailed understanding of these forces because their net effect is to fix the *relative* positions and/or velocities of different parts of macroscopic bodies, and this can be described relatively simply in terms of geometric constraints (provided we know how to handle constraints when analyzing a system's equations of motion).

2.5.1 Holonomic constraints

In §2.1.3 above we used the Lagrangian formalism to rewrite the $3N$ components of the positions $\mathbf{r}_a(t)$ of N particles in terms of an equal number of generalized coordinates q^A (with $A = 1, \cdots, 3N$) and arrived in this way at Euler-Lagrange equations expressed using the more general coordinates. Having an equal number of coordinates was required in order for the transformation between $\mathbf{r}_a$ to q^A to be invertible.

But the discussion above about constraints suggests that the requirement that we use $3N$ generalized variables is too strong, since often constraints arise as consequences of the properties of interactomic forces. There may be more than one such a constraints and each one reduces the number of independent variables needed to describe the system. In the case of a rigid body this reduction is draconian: one might start with an Avogadro's number of atoms but be left only with six independent variables (centre-of-mass position and angular orientation) once the constraints are taken into account.

Consider a set of N_c constraints the form

$$c^\alpha(q^1, q^2, \cdots, q^N, t) = c^\alpha(q, t) = 0\,, \tag{2.5.1}$$

with $\alpha = 1, \ldots, N_c$. One can often solve these equations to eliminate N_c of the variables $q^A = q^A(\hat{q}, t)$, leaving the system described by a reduced set $\hat{q}^u$ with $u = 1, \ldots, 3N - N_c$ unconstrained generalized coordinates. The number of unconstrained variables left after doing so is called the number of *degrees of freedom* in the system. If the holonomic constraints c^α do not depend on time they are called *natural*. When they are time-dependent they are often called *forced* constraint.

It must be said that it is not always possible to frame constraints as a set of implicit conditions like (2.5.1) (as is discussed in more detail in §2.5.3 below). When it is possible the constraints are said to be *holonomic*. For simplicity we start off here assuming the constraints are holonomic in this way.

It helps to keep the discussion concrete by explicitly thinking through a simple example, so first a pause to work through an example with a forced holonomic constraint.

Worked example: Bead on a rotating circular wire

Consider a small bead of mass m free to move without friction along a circular wire of radius R. This provides a well-known simple example of a forced holonomic constraint if the wire is made to move. We here work through the case where the circular wire rotates with constant angular speed ω about a vertical axis that is also a diameter of the circular wire (see Fig. 6).

For this system the constraints are easy to express if we use spherical polar coordinates, (r, θ, ϕ), for which the z axis ($\theta = 0$) is chosen to be the vertical axis of rotation. We choose the direction of this axis such that $\theta = 0$ points down. The Lagrangian for a point particle of mass m in polar coordinates is as given in §2.1.2, with kinetic energy $K = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2)$. In these coordinaes the potential energy of the mass due to the Earth's gravitational field is similarly $V = mgh$

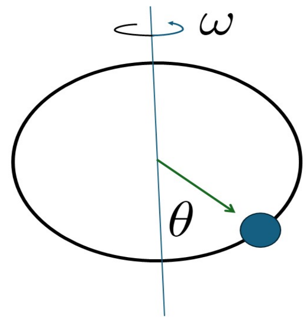


Figure 6. The geometry of a bead of mass m free to slide along a circular wire with radius R, with the wire rotating with angular speed ω about a vertical axis that is also one of its diameters.

where $h = r(1-\cos\theta)$ is the height above the bottom of the circle. In the absence of constraints the Lagrangian for the motion of the bead in these coordinates therefore is

$$L = \tfrac{1}{2}m\Big(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2\Big) - mgr\Big(1-\cos\theta\Big)\,. \tag{2.5.2}$$

To this must be added the constraint that the bead move along the wire, which fixes the radius to be constant: $r(t) = R$ for all t. The value of θ describes where the bead is on the circle and so is an unconstrained generalized coordinate. The value of ϕ sets the plane of the circular ring. If the circular wire rotates about the z axis with angular speed Ω then the constraints are:

$$r(t) = R \quad \text{and} \quad \dot\phi(t) = \Omega\,, \tag{2.5.3}$$

These constraints are simple enough to be explicitly solved so that r and ϕ can be eliminated as independent degrees of freedom: $r(t) = R$ and $\phi(t) = \Omega\,t$, where we shift the origin of ϕ so that $\phi(0) = 0$.

This leaves $\theta(t)$ as the only unconstrained degree of freedom and evaluating the Lagrangian $L(r,\theta,\phi)$ at the solution to the constraints allows the identification of the Lagrangian $L_c(\theta) = L(R,\theta,\Omega t)$ effectively governing the evolution of θ:

$$L_c = \tfrac{1}{2}mR^2\Big[\dot\theta^2 + \Omega^2\sin^2\theta\Big] - mgR\Big(1-\cos\theta\Big)\,. \tag{2.5.4}$$

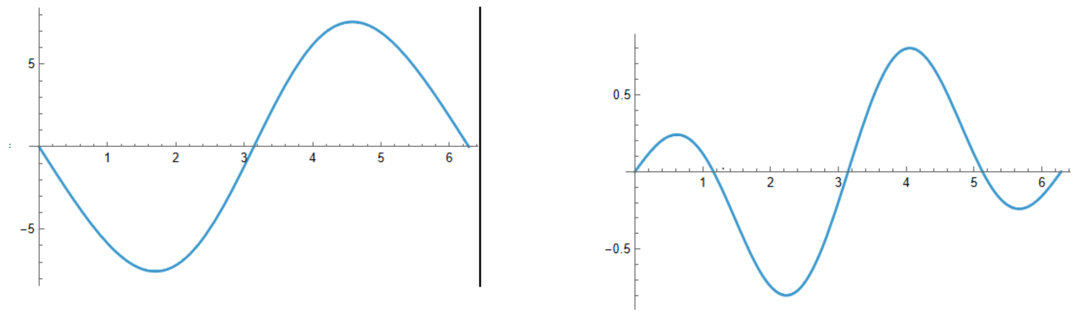
The generalized momentum for θ then becomes

$$p_\theta = \frac{\partial L_c}{\partial\dot\theta} = mR^2\,\dot\theta\,, \tag{2.5.5}$$

while the Euler-Lagrange equation for θ obtained by demanding $\int \mathrm{d}\tau\, L_c$ be extremized is

$$\ddot\theta = \left(\Omega^2\cos\theta - \frac{g}{R}\right)\sin\theta\,. \tag{2.5.6}$$

Eq. (2.5.6) shows that the character of the motion depends strongly on the relative size of the parameters Ω^2 and g/R. Time independent solutions can only arise when the right-hand side of this equation vanishes, and when $\Omega^2 < g/R$ this only happens when $\theta = 0$ or $\theta = \pi$ (see left-hand panel of Fig. (7)). Of these only $\theta = 0$ is a stable configuration because in this case small deviations $\delta\theta$ from



(a) $\Omega^2 < g/R$ (b) $\Omega^2 > g/R$

Figure 7. Plots of the right-hand side of eq. (2.5.6) in the parameter range $\Omega^2 < g/R$ (left panel) and in the parameter range $\Omega^2 > R$ (right panel).

equilibrium generate accelerations $\ddot{\theta}$ with the opposite sign (as appropriate for the restoring force in an oscillation). For θ near π the signs of $\ddot{\theta}$ and $(\theta - \pi)$ agree (indicating instability).

These physically make sense because the bead is unstable when at the top of the wire ($\theta = \pi$) but oscillates stably when displaced from the bottom of the wire ($\theta = 0$). For small deviations $|\theta| \ll 1$ near $\theta = 0$ eq. (2.5.6) approximately becomes

$$\ddot{\theta} = \left(\Omega^2 - \frac{g}{R}\right)\theta + \mathcal{O}(\theta^2)\,, \tag{2.5.7}$$

which has the same form as a simple harmonic oscillator (see §5 for more on harmonic oscillators)

$$\ddot{x} = -\omega^2 x \tag{2.5.8}$$

for which the general solution is $x(t) = A\cos(\omega t + b)$ where A and b are integration constants and ω is revealed to be the angular velocity. Consequently (2.5.7) describes simple harmonic motion with oscillation angular frequency ω given by

$$\omega = \sqrt{\frac{g}{R} - \Omega^2}\,. \tag{2.5.9}$$

This frequency approaches the frequency of a simple pendulum as $\Omega \to 0$ but vanishes when $\Omega = g/R$ (indicating the onset of an instability).

For $\Omega^2 > g/R$ two new things happen (see the right panel of Fig. 7). First, $\theta = 0$ becomes an unstable stationary point, just like $\theta = \pi$. But two new stable equilibria also arise at $\theta = \theta_\star$ with

$$\cos\theta_\star = \frac{g}{R\Omega^2}\,. \tag{2.5.10}$$

Eq. (2.5.6) implies small oscillations, $\theta = \theta_\star + \delta\theta$, around this point are approximately described by

$$\delta\ddot{\theta} = -\Omega^2\sin^2\theta_\star\delta\theta + \mathcal{O}[(\delta\theta)^2]\,, \tag{2.5.11}$$

and so have angular frequency

$$\omega = \Omega\sin\theta_\star = \sqrt{\Omega^2 - \frac{g^2}{R^2\Omega^2}}\,. \tag{2.5.12}$$

When $\Omega \gg g/R$ we have $\theta_\star \to \frac{\pi}{2}$ and $\omega \to \Omega$ corresponding to oscillations about an equilibrium with the bead at the same height as the circular wire's centre. By contrast, the oscillation frequency (2.5.12) vanishes when $\Omega \to g/R$ from above (at which point $\theta_\star \to 0$ and the stable minimum at $\theta = 0$ re-emerges for $\Omega^2 < g/R$).

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The method used in the above example explicitly solves the constraint equations to eliminate some of the variables so that the remaining variables are all unconstrained. The good news is that the Euler-Lagrange equations for these remaining unconstrained variables, $\hat{q}^u$, have the same form – eqs. (2.2.4), or $(\mathrm{d}/\mathrm{d}t)(\partial \hat{L}/\partial \dot{\hat{q}}^u) = \partial \hat{L}/\partial \hat{q}^u$ – we've been studying all along, provided that their Lagrangian is computed from the original Lagrangian by eliminating the constrained variables

$$\widehat{L}[\hat{q}, \dot{\hat{q}}, t] = L[q(\hat{q}, \dot{\hat{q}}), \dot{q}(\hat{q}, \dot{\hat{q}}), t]\,. \tag{2.5.13}$$

The equations remain unchanged because eliminating the constrained variables does not change the fact that the Lagrangian is a function of the new variables and their first derivatives (with, for instance, no dependence on higher derivatives) and so satisfies the same assumptions that led to (2.2.4).

However, as mentioned earlier, it is not always possible to formulate constraints in a way that can be solved this explicitly, so it is useful to have another way to proceed that does not rely on the ability to solve the constraints by brute force.

2.5.2 Method of Lagrange Multipliers

An important alternative approach to constrained problems is called the method of *Lagrange Multipliers*. A variant of this technique is often used when minimizing ordinary functions, as is briefly summarized in Appendix §A.2.

Suppose we wish to extremize an action $S[q(t)]$ subject to a collection of N_c constraints $c^\alpha(q, \dot{q}, t) = 0$, with $\alpha = 1, \ldots, N_c$. As always, the idea is to trade this for a different problem for which all extremizations can be done without constraints, but in this case to do so despite not being able to solve the constraints explicitly. The method of Lagrange Multipliers tells us to proceed as follows: first, one keeps *all* of the original variables q (*i.e.* temporarily ignore the constraints). Then one adds a new dynamical variable $\lambda_\alpha(t)$ for each constraint. The new variables $\lambda_\alpha(t)$ are called Lagrange Multiplier fields.

The main idea is this: the solution to the extremal problem for the original constrained action is the same as the solution to the related problem of extremizing a slightly different action:

$$S[q(t), \lambda(t)] = \int_{t_0}^{t_f} \mathrm{d}\tau \, \Big[L(q, \dot{q}, \tau) + \lambda_\alpha c^\alpha(q, \dot{q}, \tau) \Big]\,, \tag{2.5.14}$$

without imposing constraints at all. Here $L(q, \dot{q}, t)$ is the original Lagrangian ignoring the constraints and there is the usual implied sum over α. The point is that S is to be varied with respect to *all* of the q's and with respect to λ^α. Because the action is linear in λ^α the extremal equations obtained by varying them are very simple: they just impose the desired constraints $c^\alpha(q, \dot{q}, t) = 0$. The variation of the q's now receives contributions from both $S[q]$ and from the c^α but the arguments of Appendix §A.2 show why the result agrees with the result found in the constrained problem.

To see how this works it is again worth examining in detailed a simple example.

Worked example: The Atwood Machine

Consider the example of an Atwood's machine involving two masses that are connected by a taut rope of length ℓ that is wrapped around a frictionless pulley whose axis is horizontal (so the masses are hand suspended under the pulley) – see Fig. 8. The mass of the rope and pulley are neglected so gravity can be regarded as acting only on the two masses and we can ignore the kinetic energy of the rotating pulley.

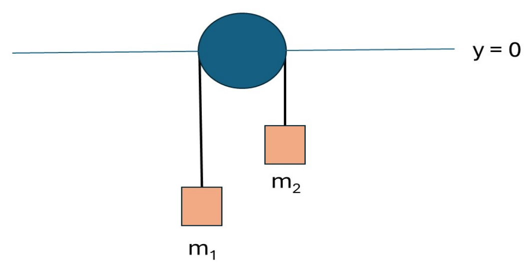

Figure 8. The geometry of the Atwood machine: two masses suspended by a rope of length ℓ wrapped around a pulley of radius R whose axis of rotation is horizontal.

In this case the masses are free to move vertically as the pulley rotates, so each has a position variable, $y_i(t)$ with $i = 1, 2$, measuring its height above a reference point, which we take to be the height of the centre of the pulley (which we can also take to be the zero of the potential energy, which is, after all, only defined up to an additive constant). Adopting the convention that positive y means 'down' – as in the figure – then the potential energy for each mass is $-m_i g\, y_i$ (up to an additive y_i-independent constant), where $g \simeq 9.8$ m/s^2 is the acceleration of gravity at the Earth's surface.

The kinetic energy of each of the masses is similarly $\frac{1}{2} m_i\, \dot{y}_i^2$, so the total kinetic and potential energies are

$$K = \tfrac{1}{2} m_1 \dot{y}_1^2 + \tfrac{1}{2} m_2 \dot{y}_2^2 \quad \text{and} \quad V = V_0 - m_1 g\, y_1 - m_2 g\, y_2\,, \tag{2.5.15}$$

where V_0 is an arbitrary constant. The Lagrangian for the entire system would then naively be $L = K - V$ and so

$$L = \tfrac{1}{2}\Big(m_1 \dot{y}_1^2 + m_2 \dot{y}_2^2\Big) + \Big(m_1 y_1 + m_2 y_2\Big) g - V_0\,. \tag{2.5.16}$$

This is 'naive' because it ignores the fact that the masses are tied together by the rope and so cannot move independently of one another (for a taut rope). Since the rope has a fixed length ℓ the positions of the two masses must satisfy the constraint

$$\ell = y_1 + y_2 + \pi R \tag{2.5.17}$$

where R is where the radius of the pulley. This system is simple enough that we can solve it explicitly, first by explicitly solving the constraint and second by using the method of Lagrange multipliers.

The constraint (2.5.17) is trivial to solve: $y_2 = -y_1 + \ell - \pi R$ and so $\dot{y}_2 = -\dot{y}_1$. From here on in we use these to eliminate y_2 from L and we denote y_1 simply by y for notational convenience. The lagrangian (2.5.16) then becomes

$$L = \tfrac{1}{2}(m_1 + m_2)\dot{y}^2 + (m_1 - m_2)gy - \tilde{V}_0\,, \tag{2.5.18}$$

where $\tilde{V}_0$ is another constant (whose value is not needed).

With these choices the generalized momentum is

$$p := \frac{\partial L}{\partial \dot{y}} = (m_1 + m_2)\dot{y} \tag{2.5.19}$$

and so the Euler-Lagrange equation (2.2.4) defining the extremum becomes

$$\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\partial L}{\partial y} = 0 \quad \text{or} \quad \ddot{y} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g\,. \tag{2.5.20}$$

This agrees with the result found by applying $\mathbf{F} = m\mathfrak{a}$ to each mass though does so without having to first identify the magnitude T of the tension on the rope. The masses experience constant acceleration whose value can be much smaller than g if the two masses are close to being equal (which in practice makes it much easier to measure the value of g more precisely).

Next solve this model using the method of Lagrange multipliers, to verify that it gives the same solution. In this case we introduce one Lagrange multiplier field $\lambda(t)$ and replace the Lagrangian (2.5.16) with the new one:

$$L_{\rm new} = \tfrac{1}{2}\Big(m_1\dot{y}_1^2 + m_2\dot{y}_2^2\Big) + \Big(m_1 y_1 + m_2 y_2\Big)g - \lambda\Big(y_1 + y_2\Big)\,, \tag{2.5.21}$$

which drops the irrelevant additive constant in L. We now vary y_1, y_2 and λ *without* imposing any constraints among them, demanding $\int \mathrm{d}\tau L_{\rm new}$ be extremized. This leads to three conditions:

$$\delta y_1: \quad -m_1\ddot{y}_1 + m_1 g - \lambda = 0 \tag{2.5.22a}$$
$$\delta y_2: \quad -m_2\ddot{y}_2 + m_2 g - \lambda = 0 \tag{2.5.22b}$$
$$\delta\lambda: \quad y_1 + y_2 = 0\,. \tag{2.5.22c}$$

Eqs. (2.5.22a) and (2.5.22b) provide an interpretation for λ. These equations express Newton's 2nd Law in the vertical direction for the two blocks if λ is the tension on the rope. Writing (as before) $y_1 = y$ and using (2.5.22c) to find $y_2 = -y$ the remaining two equations can be solved for $\ddot{y}$ and λ. The $\ddot{y}$ expression that results is

$$(m_1 + m_2)\ddot{y} + (m_2 - m_1)g = 0 \tag{2.5.23}$$

in agreement with (2.5.20). The value for λ found by eliminating $\ddot{y}$ between (2.5.22a) and (2.5.22b) similarly determines the rope's tension

$$\lambda = \frac{2m_1 m_2 g}{m_1 + m_2}\,, \tag{2.5.24}$$

in agreement with the standard result.

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Although it is good to see explicitly that new techniques agree with old ones, the Atwood machine is such a simple system that finding its solution using Lagrangian multipliers is like swatting flies with an anvil. The method of Lagrange multipliers really comes into its own in situations where solutions to the constraints are not available (where it works equally well).

2.5.3 Nonholonomic Constraints

This section clarifies how it can be that some constraints are not holonomic, since these are not just academic oddities and can arise in very simple systems. Perhaps the simplest is in the description of a systems whose surfaces roll without slipping relative to one another.

The description of rolling without slipping involves geometrical constraints – see the discussion in §1.5 – because the no-slip condition requires the surfaces of the two objects instantaneously not to move relative to one another at the point of contact. This implies a relation between the speed of motion of the centre of mass and the angular speed with which the rolling object turns; a constraint (or constraints) linear in the generalized velocities – see *e.g.* eq. (1.5.2).

In general a constraint linear in the generalized velocities has the form

$$f_A(q)\,\dot{q}^A = g(q,t)\,, \tag{2.5.25}$$

for some coefficient functions $f_A(q,t)$ and $g(q,t)$. At first sight this seems very much consistent with the holonomic form $c(q,t) = 0$ given in (2.5.1), which when differentiated with respect to time gives

$$\frac{\partial c}{\partial q^A}\,\dot{q}^A + \frac{\partial c}{\partial t} = 0\,. \tag{2.5.26}$$

However – as was also argued in another context in §1.4 (see the discussion below eq. (1.4.7)) – for general choices for f^A and g it is not in generall true that there exists a function c satisfying

$$\frac{\partial c}{\partial q^A} = f_A \quad \text{and} \quad \frac{\partial c}{\partial t} = -g\,, \tag{2.5.27}$$

because a necessary condition for this is for the following integrability conditions to be satisfied:

$$\frac{\partial f_A}{\partial q^B} = \frac{\partial f_B}{\partial q^A} \quad \text{and} \quad \frac{\partial f_A}{\partial t} = -\frac{\partial g}{\partial q^A}\,. \tag{2.5.28}$$

When (2.5.28) no longer holds then there does not exist a function $c = c(q,t)$ of the various generalized coordinates for which (2.5.25) is equivalent to (2.5.1).

For the constraint (1.5.2) describing the case of a cylinder rolling on a flat surface we have $v = \omega R$ where $v = \dot{s}$ and $\omega = \dot{\theta}$ are the rates of change of linear and angular displacement and R is the cylinder's radius. In this case the coefficients f_A and g are constants and so

(2.5.28) holds. This means that (2.5.26) can be integrated to obtain a constraint of the form $c(q,t) = 0$ that directly relates the coordinates, which by inspection is given by

$$c = s(t) - s_0 + \theta(t) R = 0\,, \tag{2.5.29}$$

where s_0 is an arbitrary integration constant. So the constraint describing cylinders rolling without slipping over a plane is holonomic.

The same is *not* true, for example, for a sphere rolling along a plane. In this case the constraints again relate how the marble rotates as a function of where its centre of mass moves in the plane, but with the new ingredient being that the direction of the motion is not fixed. The freedom to roll the sphere in any direction means that there is not a unique relationship between the position of the sphere and its angular orientation. This can be seen most simply because the sphere can be moved in a closed path such that its centre of mass returns to its initial position but the sphere arrives with a new angular orientation. This means there is does not exist a function that specifies angular orientation as a function of position (as would be possible if the constraints were holonomic).

2.6 Non-conservative Forces

Although at a fundamental level all forces do seem to be conservative it is nonetheless true that non-conservative forces do arise at a macroscopic level and this makes it useful to extend the least-action principle to incorporate these as well.

To this end suppose we have a point particle of mass m with position $\mathbf{r}(t)$ moving in the presence of a non-conservative force $\mathbf{F}$, so Newton tells us that

$$m\ddot{\mathbf{r}} = \mathbf{F}\,. \tag{2.6.1}$$

For this system we can ask what the variation of the time-integral,

$$I := \int \mathrm{d}t\, K(t), \tag{2.6.2}$$

of the kinetic energy $K = \frac{1}{2} m \dot{\mathbf{r}}^2$ is, with the result

$$\delta \int_{t_0}^{t_f} \mathrm{d}t\, K[\mathbf{r}(t)] = \int_{t_0}^{t_f} \mathrm{d}t\, m\dot{\mathbf{r}} \cdot \delta\dot{\mathbf{r}} = \Big[m\dot{\mathbf{r}} \cdot \delta\mathbf{r} \Big]_{t_0}^{t_f} - \int_{t_0}^{t_f} \mathrm{d}t\, m\ddot{\mathbf{r}} \cdot \delta\mathbf{r}\,. \tag{2.6.3}$$

Comparing this to (2.6.1) shows that $\delta I = -\delta W$ where

$$\delta W := \int_{t_0}^{t_f} \mathrm{d}t\; \mathbf{F} \cdot \delta\mathbf{r} \tag{2.6.4}$$

is the virtual work done by the applied force $\mathbf{F}$ if the particle were moved through the virtual displacement $\delta\mathbf{r}$. The displacement and work are called 'virtual' because $\delta\mathbf{r}$ here is the deviation between two paths in the integral (as opposed to the 'real' work done by the force as the particle moves along its real trajectory from $\mathbf{r}(t)$ to $\mathbf{r}(t + \delta t)$).

2.6.1 Generalized Action Principle

We see that the way to formulate the action principle for non-conservative forces is to assert that $\delta I + \delta W = 0$ where I and δW are respectively defined by (2.6.2) and (2.6.4). (Notice that (2.6.4) defines δW rather than W itself, but this is all that is needed.)

As usual, the result is much more useful when extended to multi-particle systems and expressed in terms of a more general set of generalized coordinates q^A. In this case we write $\mathbf{r}_a(t) = \mathbf{r}_a[q(t)]$ where the index a on $\mathbf{r}_a$ runs over the different particles and the index A on the generalized coordinate q^A runs over the pair $\{ai\}$ where a labels the particle and $i = x, y, z$ labels the components of vectors. In this more general formulation varying the kinetic integral I gives

$$\delta I = \left[\frac{\partial K}{\partial \dot{q}^A}\,\delta q^A\right]_{t_0}^{t_f} + \int_{t_0}^{t_f} \mathrm{d}t \left[-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}^A}\right) + \frac{\partial K}{\partial q^A}\right]\delta q^A \tag{2.6.5}$$

and using $\delta \mathbf{r}_a = (\partial \mathbf{r}_a/\partial q^A)\,\delta q^A$ implies

$$\delta W = \int_{t_0}^{t_f} \mathrm{d}t\, \mathcal{F}_A\, \delta q^A \tag{2.6.6}$$

where as usual there is an implied sum over the index A and the *generalized force* is defined by

$$\mathcal{F}_A := \sum_a \mathbf{F}_a \cdot \delta \mathbf{r}_a = \sum_a \mathbf{F}_a \cdot \frac{\partial \mathbf{r}_a}{\partial q^A}\,. \tag{2.6.7}$$

Combining these extensions to more particles and more general coordinates implies that the action principle $\delta I + \delta W = 0$ subject to the conditions $\delta q^A(t_0) = \delta q^A(t_f) = 0$ gives the following generalized equations of motion in the non-conservative case

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}^A}\right) = \frac{\partial K}{\partial q^A} + \mathcal{F}_A\,. \tag{2.6.8}$$

It is important to recognize that the generalized force components $\mathcal{F}_A$ are *not* simply the components of the applied force written in curvilinear coordinates. For example for motion in a plane described by polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ the kinetic energy of a particle would be $K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$ and so (2.6.8) specializes to the two equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(m\dot{r}\Big) = mr\dot{\theta}^2 + \mathcal{F}_r \tag{2.6.9a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(mr^2\dot{\theta}\Big) = \mathcal{F}_\theta\,. \tag{2.6.9b}$$

In this case the left-hand side of (2.6.9b) gives the rate of change of the component of angular momentum perpendicular to the plane of motion and so $\mathcal{F}_\theta$ is the net torque on the particle in this direction due to the applied forces, $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ (as opposed to the component of $\mathbf{F}$ in the $\mathbf{e}_\theta$ direction).

As mentioned earlier for general non-conservative $\mathbf{F}_a$ (and so general nonconservative $\mathcal{F}_A$) there need not exist a quantity $W[q(t)]$ whose variation gives the above expression for δW. But sometimes there is, most notably in the special case when the applied forces are conservative, and so $\mathbf{F}_a = -\nabla_a V$ for some potential energy $V = V(\mathbf{r}_1, \dots, \mathbf{r}_N)$. In this case we have

$$\delta W = \sum_a \mathbf{F}_a \cdot \delta \mathbf{r}_a = -\sum_a \nabla_a V \left(\frac{\partial \mathbf{r}_a}{\partial q^A} \right) \delta q^A = -\frac{\partial V}{\partial q^A}\, \delta q^A \tag{2.6.10}$$

which implies both $\mathcal{F}_A = -\partial V/\partial q^A$ and $\delta W = -\delta V$. Consequently the equations of motion (2.6.8) reduce to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}^A} \right) = \frac{\partial K}{\partial q^A} - \frac{\partial V}{\partial q^A}\,, \tag{2.6.11}$$

in agreement with the Euler-Lagrange equations (2.2.4) if $L = K - V$ (because $\partial V/\partial \dot{q}^A = 0$ ensures $\partial L/\partial \dot{q}^A = \partial K/\partial \dot{q}^A$).

But (2.2.4) can also sometimes apply even if the applied forces are not conservative. In particular, if it is true that

$$\mathcal{F}_A = -\frac{\partial V}{\partial q^A} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial V}{\partial \dot{q}^A} \right) \tag{2.6.12}$$

for some function $V(q, \dot{q})$ of both position *and* velocity then it is still true that (2.6.8) is equivalent to (2.2.4) once we make the identification $L = K - V$. This is not just an empty loophole, as the following important example makes explicit.

2.6.2 Charged Particles in Electromagnetic Fields

Consider a collection of particles with positions $\mathbf{r}_a(t)$, masses m_a and electric charges q_a moving in the presence of electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. In this case the forces experienced by the particles due to the electromagnetic fields are given by

$$\mathbf{F}_a = q_a \Big(\mathbf{E} + \dot{\mathbf{r}}_a \times \mathbf{B} \Big) \tag{2.6.13}$$

where there is no sum here over the particle label a. We wish to show that this force is a particular example of (2.6.12) for which $\mathbf{F}_a$ is not simply the gradient of a potential but the equations of motion can still be written in Euler-Lagrange form (2.2.4) for some choice of Lagrangian L.

To this end it is useful to write the electric and magnetic fields in terms of the electrostatic potential Φ and the vector potential $\mathbf{A}$, with

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}\,, \tag{2.6.14}$$

but we do *not* assume Φ or $\mathbf{A}$ to be independent of time. For future purposes recall that (2.6.14) does not uniquely define the potentials Φ and $\mathbf{A}$ because the electric and magnetic fields remain unchanged if the potentials are changed by a gauge transformation

$$\mathbf{A} \to \mathbf{A} + \nabla \chi \quad \text{and} \quad \Phi \to \Phi - \partial_t \chi\,, \tag{2.6.15}$$

for an arbitrary function χ.

Consider the following function defined at the position of the various charged particles:

$$V := \sum_a q_a \Big\{ \Phi[\mathbf{r}_a(t), t] - \dot{\mathbf{r}}_a \cdot \mathbf{A}[\mathbf{r}_a(t), t] \Big\} . \tag{2.6.16}$$

This function satisfies

$$\frac{\partial V}{\partial x_a} = q_a \left[\frac{\partial \Phi}{\partial x_a} - \dot{x}_a \frac{\partial A_x}{\partial x_a} - \dot{y}_a \frac{\partial A_y}{\partial x_a} - \dot{z}_a \frac{\partial A_z}{\partial x_a} \right] , \tag{2.6.17}$$

and similarly for $\partial V/\partial y_a$ and $\partial V/\partial z_a$. It also satisfies

$$\frac{\partial V}{\partial \dot{x}_a} = -q_a A_x \, , \quad \frac{\partial V}{\partial \dot{y}_a} = -q_a A_y \quad \text{and} \quad \frac{\partial V}{\partial \dot{z}_a} = -q_a A_z \, , \tag{2.6.18}$$

where in all of these partial derivatives t is held fixed. The last equations also imply

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial V}{\partial \dot{x}_a} \right) = -q_a \frac{\mathrm{d}A_x}{\mathrm{d}t} \, , \quad \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial V}{\partial \dot{y}_a} \right) = -q_a \frac{\mathrm{d}A_y}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial V}{\partial \dot{z}_a} \right) = -q_a \frac{\mathrm{d}A_z}{\mathrm{d}t} \, , \tag{2.6.19}$$

where $\mathbf{A}[\mathbf{r}_a(t), t]$ has both explicit time dependence and implicit time-dependence through its dependence on the positions of the (possibly moving) particles $\mathbf{r}_a(t)$, so

$$\frac{\mathrm{d}\mathbf{A}[\mathbf{r}_a, t]}{\mathrm{d}t} = \frac{\partial \mathbf{A}}{\partial t} + \dot{x}_a \frac{\partial \mathbf{A}}{\partial x_a} + \dot{y}_a \frac{\partial \mathbf{A}}{\partial y_a} + \dot{z}_a \frac{\partial \mathbf{A}}{\partial z_a} . \tag{2.6.20}$$

Combining the above expressions shows that

$$\begin{aligned} -\frac{\partial V}{\partial x_a} + \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial V}{\partial \dot{x}_a} \right) &= -q_a \left[\frac{\partial \Phi}{\partial x_a} - \dot{x}_a \frac{\partial A_x}{\partial x_a} - \dot{y}_a \frac{\partial A_y}{\partial x_a} - \dot{z}_a \frac{\partial A_z}{\partial x_a} \right] \\ &\qquad -q_a \left[\frac{\partial A_x}{\partial t} + \dot{x}_a \frac{\partial A_x}{\partial x_a} + \dot{y}_a \frac{\partial A_x}{\partial y_a} + \dot{z}_a \frac{\partial A_x}{\partial z_a} \right] \\ &= q_a \left[- \left(\frac{\partial \Phi}{\partial x_a} + \frac{\partial A_x}{\partial t} \right) + \dot{y}_a \left(\frac{\partial A_y}{\partial x_a} - \frac{\partial A_x}{\partial y_a} \right) + \dot{z}_a \left(\frac{\partial A_z}{\partial x_a} - \frac{\partial A_x}{\partial z_a} \right) \right] \\ &= q_a \Big[E_x + \dot{y}_a B_z - \dot{z}_a B_y \Big]_{\mathbf{r} = \mathbf{r}_a} \end{aligned} \tag{2.6.21}$$

which is just the x component of the force law (2.6.13) once (2.6.14) is used. Repeating this exercise for the y and z components verifies that the motion of a charged particle in an electromagnetic field provides an example of a nonconservative velocity-dependent force for which (2.6.12) applies and so the equations of motion can be described by a Lagrangian.

We see in this way that the Lagrangian describing a collection of charged particles interacting with electromagnetic fields is:

$$L = K - V = \sum_a \Big[\tfrac{1}{2} m_a \dot{\mathbf{r}}^2 + q_a \Big(-\Phi + \dot{\mathbf{r}}_a \cdot \mathbf{A} \Big) \Big] \, , \tag{2.6.22}$$

where for each term of the sum the fields are evaluated at the positions of the corresponding particle. Notice that the electromagnetic fields introduce a linear dependence on $\dot{\mathbf{r}}_a$ and as a result the expression for the generalized momenta in this type of system become modified:

$$\mathbf{p}_a := \frac{\partial L}{\partial \dot{\mathbf{r}}_a} = m_a \dot{\mathbf{r}}_a + q_a \mathbf{A}(\mathbf{r}_a, t) \,. \tag{2.6.23}$$

Notice that (2.6.23) is *not* invariant under the gauge transformation (2.6.15) and so is not in itself physical, though the velocity $\dot{\mathbf{r}}_a$ is.

The conserved energy in this case is similarly

$$E = \sum_a \dot{\mathbf{r}}_a \cdot \frac{\partial L}{\partial \dot{\mathbf{r}}_a} - L = \sum_a \Big(\tfrac{1}{2} m_a \dot{\mathbf{r}}^2 + q_a \Phi \Big) \,, \tag{2.6.24}$$

and so is independent of $\mathbf{A}$. Although E is also not gauge invariant the integral $\int \mathrm{d}t E$ is because under (2.6.15) E shifts by a total time derivative.

Worked example: Charged particle in a constant magnetic field

Consider the case of a charged particle with mass m and charge q moving in a constant magnetic field $\mathbf{B}$ and vanishing electric field $\mathbf{E} = 0$. For this type of field we can choose

$$\Phi = 0 \qquad \text{and} \qquad \mathbf{A} = \tfrac{1}{2} \mathbf{B} \times \mathbf{r} \,. \tag{2.6.25}$$

It is convenient to work with cylindrical polar coordinates $\{\rho, \varphi, z\}$ which is obtained from Cartesian coordinates $\{x, y, z\}$ using

$$x = \rho \cos\varphi \quad \text{and} \quad y = \rho \sin\varphi \,, \tag{2.6.26}$$

and so the kinetic energy becomes

$$K = \tfrac{1}{2} m \Big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \Big) = \tfrac{1}{2} m \Big(\dot{\rho}^2 + \rho^2 \dot{\varphi}^2 + \dot{z}^2 \Big) \,. \tag{2.6.27}$$

In this basis the vector potential has components

$$\mathbf{A} = A_\rho \, \mathbf{e}_\rho + A_\varphi \, \mathbf{e}_\varphi + A_z \, \mathbf{e}_z \,, \tag{2.6.28}$$

where $\mathbf{e}_\rho = \mathbf{e}_x \cos\varphi + \mathbf{e}_y \sin\varphi$ and $\mathbf{e}_\varphi = -\mathbf{e}_x \sin\varphi + \mathbf{e}_y \cos\varphi$ and its curl is given by

$$\nabla \times \mathbf{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right) \mathbf{e}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \mathbf{e}_\varphi + \frac{1}{\rho} \left(\frac{\partial (\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi} \right) \mathbf{e}_z \,. \tag{2.6.29}$$

In particular the choice $\mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{r}$ corresponds to

$$A_\rho = A_z = 0 \qquad \text{and} \qquad A_\varphi = \tfrac{1}{2} B \rho \,, \tag{2.6.30}$$

since when this is used in (2.6.29) it gives $\mathbf{B} = \nabla \times \mathbf{A} = B \, \mathbf{e}_z$.

The Lagrangian for the charged particle obtained for the charged particle from (2.6.22) then is

$$L = \tfrac{1}{2} m \Big(\dot{\rho}^2 + \rho^2 \dot{\varphi}^2 + \dot{z}^2 \Big) + \tfrac{1}{2} q B \rho^2 \dot{\varphi} \,. \tag{2.6.31}$$

Because z and φ only appear in L differentiated they are both ignorable coordinates and so their equations of motion state that their generalized momenta are constants. For motion parallel to the magnetic field this states

$$\dot{p}_z = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}}\right) = m\ddot{z} = 0 \tag{2.6.32}$$

and so the particle does not accelerate in this direction and so simply remembers its initial conditions: $z(t) = z_0 + \dot{z}_0(t - t_0)$.

For motion in the φ direction the equations of motion state

$$\dot{p}_\varphi = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(m\rho^2\dot{\varphi} + \tfrac{1}{2}qB\rho^2\Big) = 0\,, \tag{2.6.33}$$

and so even though the angular momentum in the direction of the magnetic field, $m\rho^2\dot{\varphi}$, is not constant there still is a conserved quantity (2.6.33). Integrating we find

$$\rho^2\Big(m\dot{\varphi} + \tfrac{1}{2}qB\Big) = J\,. \tag{2.6.34}$$

The radial equation is $\dot{p}_\rho = \partial L/\partial\rho$ where $p_\rho = \partial L/\partial\dot{\rho} = m\dot{\rho}$, and so

$$m\ddot{\rho} = m\rho\dot{\varphi}^2 + qB\rho\dot{\varphi}\,. \tag{2.6.35}$$

The radial equation shows that ρ can be a constant for all times but only if the angular speed satisfies

$$\dot{\varphi} = -\frac{qB}{m} \qquad \text{(constant } \rho)\,, \tag{2.6.36}$$

whose magnitude $\omega_c = |qB/m|$ is known as the *cyclotron frequency*. For this choice the integration constant J is given by

$$J = J_c = -\tfrac{1}{2}qB\rho^2 \qquad \text{(constant } \rho)\,. \tag{2.6.37}$$

In general the motion of ρ is found by eliminating $\dot{\varphi}$ using (2.6.34), which allows the radial equation to be written

$$m\ddot{\rho} = \rho\dot{\varphi}(m\dot{\varphi} + qB) = \frac{\rho}{m}\left(\frac{J}{\rho^2} - \tfrac{1}{2}qB\right)\left(\frac{J}{\rho^2} + \tfrac{1}{2}qB\right)\,. \tag{2.6.38}$$

Multiplying this through by $\dot{\rho}$ allows it to be expressed as a conservation law, $\mathrm{d}E/\mathrm{d}t = 0$, where

$$E = \tfrac{1}{2}m\dot{\rho}^2 + V_{\text{eff}}(\rho) \quad \text{with} \quad V_{\text{eff}}(\rho) := \frac{1}{2m}\left(\frac{J^2}{\rho^2} + \tfrac{1}{4}q^2B^2\rho^2\right)\,. \tag{2.6.39}$$

For any finite energy E the motion is restricted to a limited range of r because the potential $V_{\text{eff}}(\rho)$ – see Fig. 9 – climbs to infinity both as $\rho \to 0$ and as $\rho \to \infty$. The potential has a minimum when $\rho^2 = \rho^2_{\text{min}} := |2J/qB|$, at which point

$$V_{\text{eff}}(\rho_{\text{min}}) = \frac{q^2B^2\rho^2_{\text{min}}}{4m} = \left|\frac{qBJ}{2m}\right|\,. \tag{2.6.40}$$

The energy is bounded below by this value and describes circular orbits with $\dot{\rho} = 0$ when $E = V_{\text{eff}}(\rho_{\text{min}})$. For larger values of energy the radius is not zero but it oscillates around the circular value between the two turning points, $\rho_\pm$, defined by the condition $V_{\text{eff}}(\rho_\pm) = E$.

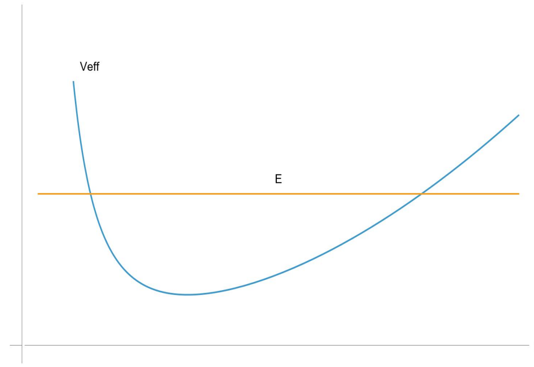

Figure 9. Sketch of the effective potential appearing in (2.6.39) governing the radial motion of a charged particle in a constant magnetic field.

Part of what makes the description of this motion complicated is that we have not yet made any particular choice about where the origin of polar coordinates should be. For a constant magnetic field there is no preferred origin of coordinates within the x-y plane and so we are always free to choose our coordinates in a convenient way. In particular, we can always choose the origin of coordinates so that the initial radial velocity is zero $\dot{\rho}(t_0) = 0$. Having done so we are also always free to move the origin closer to or further from the particle's initial position and thereby can adjust $\rho(t_0)$ to take any value we like.

A particularly convenient choice is to choose the initial radius to be $\rho_0 = \rho(t_0) = \rho_{\rm min}$, in which case $J/\rho_0^2 = \frac{1}{2}qB$. The sign of $\dot{\varphi}_0$ and J can also be changed by placing the origin the same distance away from the initial position but on the other side of the object and so it is always possible to arrange the initial conditions so that $J/\rho_0^2 = -\frac{1}{2}qB$ and so $\dot{\varphi}_0 = -qB/m$. But these choices then ensure that $\dot{\rho} = 0$ for all time (compare with (2.6.36)). This conclusion is also clear because the choice $\rho_0 = \rho_{\rm min}$ guarantees $E = V_{\rm eff}(\rho_0)$ and $V'_{\rm eff}(\rho_0) = 0$ and so conservation of E and expression (2.6.39) imply $\dot{\rho}$ must vanish.

With the convenient choice of origin we learn both $\dot{\rho} = 0$ while $\dot{z}$ and $\dot{\varphi} = -qB/m$ are constants for all t: the particle moves along a circle (if $\dot{z} = 0$) or along a helix (if $\dot{z} \neq 0$). The radius of the circle (or helix) is an initial condition and corresponds to the initial choice of linear velocity since $v_0 = |\rho_0 \dot{\varphi}_0| = \omega_c \rho_0$. The apparently more complicated behaviour of ρ for other choices of coordinates are simply the more complicated description of uniform circular motion to be expected if one doesn't choose coordinates whose origin lies at the circle's centre.

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3 Noninertial Reference Frames

We normally work in an inertial frame when analyzing a system's motion because inertial frames are the only ones for which Newton's second law is true. But nobody can stop you from translating the results found in an inertial frame over to another non-inertial one, and there can be circumstances where this is genuinely useful to do (such as in our discussion of polar coordinates in §1.2.1). Prominent among these are situations where we wish to describe circumstances using coordinates adapted to observers located on the surface of the Earth, because the Earth rotates about its axis and also orbits the Sun (which in turn orbits other things).

The following sections consider in turn two possible sources of non-inertial reference frame: (*i*) when the origin of coordinates moves with varying speed relative to an inertial frame, or (*ii*) when the basis vectors $\mathbf{e}_i$ rotate in a time-dependent way (for fixed origin). For now we do not combine both at once, but this is done later on in §4.

3.1 Accelerated reference frames

Consider first the situation where the origin of coordinates for a reference frame $\widetilde{O}$ is displaced relative to an inertial reference frame O by a time-dependent vector $\mathbf{a}(t)$ (see Fig. 5) but without changing the orientation of the basis vectors $\mathbf{e}_i$. In this case vector addition shows that the position $\mathbf{r}(t)$ and $\tilde{\mathbf{r}}(t)$ of a particle relative to these two reference frames are related by

$$\mathbf{r}(t) = \tilde{\mathbf{r}}(t) + \mathbf{a}(t)\,. \tag{3.1.1}$$

We know that in the inertial frame Newton's laws state that

$$m\ddot{\mathbf{r}} = \mathbf{F} \tag{3.1.2}$$

where $\mathbf{F}$ is the net force applied to the particle. This implies $\tilde{\mathbf{r}}$ satisifies

$$m\ddot{\tilde{\mathbf{r}}} = \mathbf{F} - m\ddot{\mathbf{a}}\,, \tag{3.1.3}$$

showing how the acceleration of reference frame appears as a *fictitious force* $\mathbf{F}_{\rm fic} = -m\ddot{\mathbf{a}}$, when written on the right-hand side of the equations of motion.

How does one determine whether any particular force appearing in Newton's laws is fictitious in this way? A smoking gun that is useful to keep track of is the proportionality of the force to m. This is generic for any fictitious force because it is being designed to produce a specific acceleration: the acceleration that defines the non-inertial frame.

As mentioned above, we know in practice that we cannot be in an inertial frame when sitting on the surface of the Earth because of the Earth's rotation, its orbit around the Sun, the Sun's orbit within the galaxy and so on. So why do Newton's laws seem to apply so well in practice in everyday experience?

To answer this we can estimate the size of each source of fictitious acceleration we should be experiencing (and can compare them to the acceleration of gravity, $g = |\mathbf{g}| \simeq 9.8$ m/s^2 as a useful benchmark. To do so we use the useful formula from elementary physics courses for the centripetal acceleration $a_c = v^2/r = \Omega^2 r$ of an object moving in a circular orbit of radius r when moving with speed v (or angular speed $\Omega = v/r$).

- **Earth's rotation:** Taking the Earth's rotation period to be 1 sidereal day (around 86,000 seconds) implies $\Omega_{\rm rot} \simeq 7.9 \times 10^{-5}$/s so using the mean radius of the Earth of around $R_\oplus \simeq 6400$ km implies $a_c \simeq 0.03$ m/s^2. Consequently $a_c/g \simeq 3 \times 10^{-3}$.
- **Earth's orbit around the Sun:** Taking the orbital period to be 1 sidereal year (3.1556×10^7 seconds) implies $\Omega_{\rm orb} \simeq 1.99 \times 10^{-7}$/s and using the radius of the orbit is $R_{ES} = 1$ AU$= 1.4960 \times 10^8$ km gives $a_c \simeq 0.0059$ m/s^2 and so $a_c/g \simeq 6 \times 10^{-4}$.

These estimates show that two things help make Newton's laws work well in practice, even for physicists trapped on the surface of the Earth. First, the fictitious accelerations implied by the Earth's motion are very small. The largest (due to the Earth's rotation - more about which below) is a correction that contributes less than a percent of the acceleration due to gravity. The fictitious acceleration due to the Earth's motion about the Sun is in principle only a bit smaller.

The second reason is specific to motion due to gravity, and so is relevant to the Earth's orbit around the Sun but not to the Earth's rotation. It arises because approximately constant gravitational fields only affect how the centre of mass of a group of objects moves and completely drops out of Newton's Laws for their relative motion (along the lines seen in §1.2.1 – see the discussion following eq. (1.2.8)). From this perspective it is only *tidal forces* – *i.e.* the *change* in *e.g.* the Sun's gravitational field across the Earth – that matter in practice for describing the motion of objects on the Earth's surface, but these are suppressed by an additional factor of $R_\oplus/R_{ES} \simeq 4 \times 10^{-5}$ relative to the orbital accelerations described above.

3.1.1 Principle of Equivalence

The force of gravity itself, $\mathbf{F}_g = m\mathbf{g}$, is also proportional to m: should this also be regarded as a fictitious force?

The first reaction to this question is to answer 'yes'. We sit on the surface of the Earth and the Earth is both rotating about its axis once a day and travelling in an orbit around the Sun, which itself follows an orbit inside our local galaxy. We certainly do not move at constant velocity and so we cannot be in an inertial reference frame. We should expect to be seeing fictitious forces.

The second reaction to this question is to answer 'no'. We can – and will, in §3.3 – compute the fictitious forces associated with the Earth's rotation and will argue why these are likely to be the largest non-inertial effects. Although fictitious forces exist they do not have the property of universally pointing down, and so are not responsible for the force $\mathbf{F}_g$.

The third reaction to this question is to answer 'yes' again. Although it goes beyond the scope of these notes to show in detail, the synthesis of relativity with gravity contained in General Relativity shows how gravitational fields are described by the curvature of space and time and freely falling trajectories provide the analogs of locally inertial frames. These frames are described by geodesics in the curved geometry of spacetime. On the Earth's surface we are not freely falling because the Earth itself stops us from following gravity's urgings to fall towards its centre and this provides another reason why our reference frame is not inertial. (A bit more about this point of view is given in §6.2.2 below.)

From this point of view we would not be inertially moving even if the Earth were not rotating or moving at all. In the end the force $\mathbf{F}_g$ *can* be regarded as a fictional force, due to the non-inertial motion implied by being held at rest at the Earth's surface and thereby being kept from freely falling. This situation is described by Einstein's 'elevator' thought experiment meant to emphasize the 'principle of equivalence': the behaviour of bodies on the Earth's surface in the presence of the gravitational force $\mathbf{F}_g = m\mathbf{g}$ is identical to what would be seen inside a windowless elevator in empty space (without the Earth) that is pulled upward with an acceleration $\ddot{\mathbf{a}} = -\mathbf{g}$ (*c.f.* eq. (3.1.3)).

Sometimes this thought experiment is taken to mean that there is a sense in which gravity really is not a force at all, but this is very misleading. It is true that the effects of a *constant* gravitational field $\mathbf{g}$ can be mimicked by accelerating one's reference frame by a constant acceleration $\ddot{\mathfrak{a}} = -\mathfrak{g}$. But the gravitational field outside of the Earth is *not* a constant - in reality it is given by a radially pointing inverse-square law: $\mathbf{F} = -(GMm/r^2)\,\mathbf{e}_r$, and so for a spherical Earth it points at the Earth's centre. This means it points in slightly different directions at different points above the Earth's surface. It is this differential change in the gravitational field – what are called 'tidal forces' – that are not fictious forces, and so are described by the gravitational field.

3.2 Rotating Reference Frames

This section describes the rotating non-inertial reference frames that are the most practical use for observers riding a rotating celestial orb. A first step in this direction was already done when discussing rotation symmetries in §1.6 and §2.3.4 in which it was shown that an infinitesimal rotation $R_{ij} = \delta_{ij} + \Theta_{ij}$ defines a vector $\boldsymbol{\Theta}$ through the relation $\Theta_{ij} = \epsilon_{ijk}\Theta_k$ (where ϵ_{ijk} is the Levi-Civita symbol, some of whose properties are described in §A.3.1).

In matrix form – see (2.3.20) – this says

$$\begin{pmatrix} 0 & \Theta_{xy} & \Theta_{xz} \\ \Theta_{yx} & 0 & \Theta_{yz} \\ \Theta_{zx} & \Theta_{zy} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Theta_z & -\Theta_y \\ -\Theta_z & 0 & \Theta_x \\ \Theta_y & -\Theta_x & 0 \end{pmatrix} . \tag{3.2.1}$$

Some physical intuition can be developed by comparing this to the rotation matrices describing *e.g.* rotations through an angle $\delta\theta$ about the x-axis:

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\delta\theta) & \sin(\delta\theta) \\ 0 & -\sin(\delta\theta) & \cos(\delta\theta) \end{pmatrix} \simeq I + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta\theta \\ 0 & -\delta\theta & 0 \end{pmatrix} + \mathcal{O}\left[(\delta\theta)^2\right] , \tag{3.2.2}$$

which agrees with $R_{ij} = \delta_{ij} + \Theta_{ij}+$(higher order) and (3.2.1) for the special case $\Theta_x = \delta\theta$ and $\Theta_y = \Theta_z = 0$. Nonzero Θ_y and Θ_z are similarly related to infinitesimal rotations about the y and z axes.

More generally, any infinitesimal rotation is defined by a rotation axis (which specifies a direction $\mathbf{n}$, for some unit vector $\mathbf{n}$) and an infinitesimal rotation angle Θ and these together are the information incorporated in the vector $\boldsymbol{\Theta} = \Theta\,\mathbf{n}$. We here adopt the right-hand rule for which the vector $\mathbf{n}$ points along the thumb if the fingers of the right hand are moved in the direction of the rotation. As shown in detail in §2.3.4 such a rotation transforms a specific vector $\mathbf{V}$ to $\mathbf{V} + \delta\mathbf{V}$ where

$$\delta\mathbf{V} = \boldsymbol{\Theta} \times \mathbf{V}\,. \tag{3.2.3}$$

This is particularly clear for the position vector itself, $\mathbf{r}$, measured from an origin of coordinates chosen to lie on the rotation axis. In this case $\delta\mathbf{r} = \boldsymbol{\Theta}\times\mathbf{r}$ has the right magnitude ($\delta r = \Theta\, r \sin\theta$, where θ is the angle between $\boldsymbol{\Theta}$ and $\mathbf{r}$) and (using the right-hand rule for the cross product) points in the correct direction (perpendicular to both $\mathbf{r}$ and $\boldsymbol{\Theta}$), as shown in Fig. 10. But it is equally true for *any* vector, as its derivation in §2.3.4 makes clear.

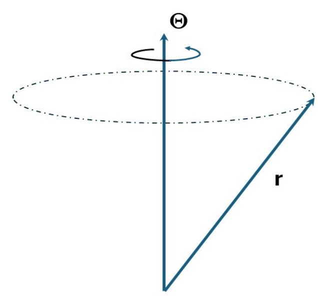

Figure 10. The geometry of the rotation of a vector $\mathbf{r}$ about an angle and axis defined by the rotation vector $\boldsymbol{\Theta}$.

If the above infinitesimal rotation occurs over a small time δt and the infinitesimal rotation is written $\Theta = \delta\theta$ to emphasize its small size then the limit $\Omega = \lim_{\delta t\to 0}(\delta\theta/\delta t)$ defines the instantaneous angular speed of the rotation and $\boldsymbol{\Omega} = \lim_{\delta t\to 0}(\boldsymbol{\Theta}/\delta t)$ is the rotation's instantaneous angular velocity, with $\Omega = |\boldsymbol{\Omega}|$. If the vector $\mathbf{V}$ already has an intrinsic time dependence, which we denote by partial derivatives $\partial\mathbf{V}/\partial t$, then the addition of the time-dependence due to the rotation implies its total rate of change is

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \frac{\partial\mathbf{V}}{\partial t} + \boldsymbol{\Omega}\times\mathbf{V}\,. \tag{3.2.4}$$

In particular, a vector satisfies

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \boldsymbol{\Omega}\times\mathbf{V} \qquad \text{(pure rotation)}\,. \tag{3.2.5}$$

if and only if the vector $\mathbf{V}$ has no intrinsic time dependence in a rotating reference frame with angular velocity $\boldsymbol{\Omega}$.

Worked example: Charged particle in a constant magnetic field (again)

As a simple example of the above discussion consider a charged particle of mass m and charge q that is free to move in a region containing a constant magnetic field $\mathbf{B}$ (see also the worked example below (2.6.23)).

The Lorentz force acting on such a particle is given by $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, where $\mathbf{v}(t)$ is the particle's instantaneous velocity. Newton's second law is to be solved for the particle's motion, and for the Lorentz force this gives

$$m\,\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\mathbf{v} \times \mathbf{B}\,. \tag{3.2.6}$$

This can be immediately solved because it has precisely the form (3.2.5) for the special case where

$$\boldsymbol{\Omega} = -\frac{q\mathbf{B}}{m}\,. \tag{3.2.7}$$

This implies that $\mathbf{v}$ must be a constant if written in a rotating reference frame whose angular velocity is given by (3.2.7). The motion therefore consists of helical motion about an axis parallel to the applied field $\mathbf{B}$ with angular frequency $\omega_c = qB/m$, precisely as was found previously (see the worked example below (2.6.23)).

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3.3 Motion at the Earth's Surface

Consider now the motion of an object O that moves near the Earths' surface and whose position vector relative to the Earth's centre is $\mathbf{r}$. We wish to compare how this motion appears to two observers; one of which is inertial and the other of which rotates with the Earth's rotation. Because we wish only to follow the effects of rotation (as opposed to the translational motion described in §3.1 above) we imagine both observers to define their origins of coordinates at the Earth's centre.[8]

Because O is carried along by the Earth's rotation its velocity $\mathrm{d}\mathbf{r}/\mathrm{d}t$ satisfies (3.2.4):

$$\mathbf{v} := \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\partial\mathbf{r}}{\partial t} + \boldsymbol{\Omega} \times \mathbf{r}\,, \tag{3.3.1}$$

where $\partial\mathbf{r}/\partial t$ describes any time-dependence of the object's position due to other causes besides the rotation of the Earth. To identify the fictitious forces due to Earth's rotation seen by an observer with coordinates adapted to the Earth's surface we differentiate (3.3.1), assuming the Earth's angular velocity $\boldsymbol{\Omega}$ is time-independent. This gives

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\partial}{\partial t}\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right) + \boldsymbol{\Omega} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\partial^2\mathbf{r}}{\partial t^2} + 2\boldsymbol{\Omega} \times \frac{\partial\mathbf{r}}{\partial t} + \boldsymbol{\Omega} \times \Big(\boldsymbol{\Omega} \times \mathbf{r}\Big)\,, \tag{3.3.2}$$

in which the second equality uses (3.3.1) again. The final result is also what is obtained if we apply (3.2.4) directly to $\mathbf{v}$.

[8] We return to simultaneous combinations of translational and rotational motion in §4.

Suppose the particle situated at O moves subject to gravity, $\mathbf{F}_g = m\mathbf{g}$, and some other mechanical force $\mathbf{F}$. In the inertial frame at the centre of the Earth Newton's Law states

$$m\,\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{F} + m\mathbf{g}\,, \tag{3.3.3}$$

and so using (3.3.2) in this implies the motion as seen by an observer using coordinates adapted to rotate with the Earth has an apparent acceleration

$$\frac{\partial^2\mathbf{r}}{\partial t^2} = \frac{\mathbf{F}}{m} + \mathbf{g} - \boldsymbol{\Omega}\times\Big(\boldsymbol{\Omega}\times\mathbf{r}\Big) - 2\,\boldsymbol{\Omega}\times\frac{\partial\mathbf{r}}{\partial t}\,. \tag{3.3.4}$$

This reveals two Ω-dependent fictitious forces due to the Earth's rotation. The one quadratic in Ω and depending on particle position $\mathbf{r}$ is called the *centrifugal force* while the one linear in Ω and proportional to the apparent particle velicity $\partial\mathbf{r}/\partial t$ is called the *Coriolis force*.

Numerically, the Earth's angular speed is 2π radians per sidereal day (86,164 seconds), and so $\Omega = 7.292 \times 10^{-5}$/s. Neglecting deviations from spherical shape the Earth's mean radius is $R_\oplus = 6,371$ km and so the relative size of the mass-independent accelerations in (3.3.4) are

$$g = 9.807\ \mathrm{m/s}^2\,, \qquad \Omega^2 R_\oplus = 0.03388\ \mathrm{m/s}^2\,, \qquad \Omega v = 7.292\times 10^{-5}\ \mathrm{m/s}^2 \left(\frac{v}{1\ \mathrm{m/s}}\right)\,. \tag{3.3.5}$$

Centrifugal and Coriolis effects are clearly subdominant to g, but can be important in some circumstances.

3.3.1 Centrifugal force

The centrifugal force can be rewritten using the vector identity (A.3.14) to give

$$\begin{aligned} -\boldsymbol{\Omega}\times\Big(\boldsymbol{\Omega}\times\mathbf{r}\Big) &= \Omega^2\mathbf{r} - (\boldsymbol{\Omega}\cdot\mathbf{r})\,\boldsymbol{\Omega} = \Omega^2 r(\mathbf{e}_r - \mathbf{e}_z\,\cos\theta) \\ &= \Omega^2 r\sin\theta(\cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y) \\ &= \Omega^2 r\sin\theta(\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta)\,, \end{aligned} \tag{3.3.6}$$

where we write $\boldsymbol{\Omega} = \Omega\,\mathbf{e}_z$ where $\mathbf{e}_z$ points up the positive z axis towards the North Pole in coordinates with origin at the Earth's centre and the equator in the x-y plane. It is also convenient to use spherical polar coordinates, $\{r, \theta, \phi\}$ in this frame, with basis unit vectors given explicitly as in (1.2.13), (1.2.14) and (1.2.15). In terms of these the particle at O is given by $\mathbf{r} = r\,\mathbf{e}_r$ so the angle ϕ describes the particle's longitude and θ describes its *co-latitude* ($\theta = 90^o - \ell$, if ℓ is the latitude). Eq. (3.3.6) then follows, and shows the centrifugal force is perpendicular to the axis of rotation and directed radially away from the axis.

Worked example: Plumb bobs and the local vertical

A simple implication of the fictitious centrifugal force on Earth is its role in causing a mass suspended from a rope to not point precisely in a vertical direction. For a static mass rotating with the Earth eq. (3.3.4) predicts the force of the rope acting on the mass must balance the sum of the gravitational and centrifugal forces: $\mathbf{F} = -m[\mathbf{g} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})]$.

Using spherical polar coordinates described above we have $\mathbf{g} = -g\,\mathbf{e}_r$, while $\boldsymbol{\Omega} = \Omega\,\mathbf{e}_z$ and $\mathbf{r} = R_\oplus\,\mathbf{e}_r$. The rope hangs in the direction $-\mathbf{F}$ and so is proportional to the 'effective' gravitational acceleration:

$$\mathbf{g}_{\rm eff} := \mathbf{g} + \Omega^2\mathbf{r} - (\boldsymbol{\Omega}\cdot\mathbf{r})\,\boldsymbol{\Omega} = (-g + \Omega^2 R_\oplus \sin^2\theta)\,\mathbf{e}_r + \Omega^2 R_\oplus \sin\theta\cos\theta\,\mathbf{e}_\theta\,, \tag{3.3.7}$$

which uses (3.3.6) to simplify the right-hand side. This shows $\mathbf{g}_{\rm eff}$ has both vertical and horizontal components (respectively defined as pointed towards the Earth's centre and tangent to its surface in a Southerly direction) of size

$$g_v := -\mathbf{g}_{\rm eff}\cdot\mathbf{e}_r = g - \Omega^2 R_\oplus \sin^2\theta \qquad \text{and} \qquad g_h := \mathbf{g}_{\rm eff}\cdot\mathbf{e}_\theta = \Omega^2 R_\oplus \sin\theta\cos\theta\,. \tag{3.3.8}$$

The rope therefore hangs at an angle ξ from the vertical, towards the south by an angle

$$\xi \simeq \tan\xi = \frac{g_h}{g_v} = \frac{\Omega^2 R_\oplus \sin\theta\cos\theta}{g - \Omega^2 R_\oplus \sin^2\theta} \simeq \frac{\Omega^2 R_\oplus}{g}\,\sin\theta\cos\theta\,. \tag{3.3.9}$$

This vanishes at the North and South Poles and at the equator, and is at its largest at 45^o latitude, where it is 0.002 radians (or about 7 minutes of arc). Since ξ changes sign once $\theta > \frac{\pi}{2}$ the rope hangs a bit to the south of vertical in the northern hemisphere and a bit north of the vertical in the southern hemisphere.

The total magnitude of the apparent acceleration of gravity is

$$g_{\rm eff} = \sqrt{g^2 - (2g - \Omega^2 R_\oplus)\Omega^2 R_\oplus \sin^2\theta} \simeq g - \Omega^2 R_\oplus \sin^2\theta + \mathcal{O}[(\Omega^2 R_\oplus)^2/g]\,, \tag{3.3.10}$$

and so in particular $g_{\rm eff} = g$ at the North and South poles but $g_{\rm eff} \simeq g - \Omega^2 R_\oplus$ at the equator. This implies a difference of about 3.4 cm/s^2 between the pole and the equator (in the idealization of a perfectly spherical Earth). In reality the Earth is slightly oblate and bulges out at the equator, which amplifies the difference of $g_{\rm eff}$ between poles and equator to more like 5.2 cm/s^2.

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3.3.2 Coriolis force

The Coriolis force is only relevant for moving objects due to the explicit factor of $\partial\mathbf{r}/\partial t$ in its definition. To explore its implications we revisit the hanging plumb bob, but now allow it to oscillate about its equilibrium position.

Worked example: Foucault's pendulum

Suppose we take the plumb bob described above and displace it from its equilibrium position and let it oscillate. In that case the motion is described by (3.3.4)

$$\frac{\partial^2\mathbf{r}}{\partial t^2} = \frac{\mathbf{F}}{m} + \mathbf{g} - \boldsymbol{\Omega}\times\Big(\boldsymbol{\Omega}\times\mathbf{r}\Big) - 2\,\boldsymbol{\Omega}\times\frac{\partial\mathbf{r}}{\partial t}\,, \tag{3.3.11}$$

where the tension $\mathbf{F}$ acts to keep the motion in an arc of radius L (the pendulum's length) in a vertical plane. The pendulum's equilibrium position satisfies

$$\frac{\mathbf{F}}{m} + \mathbf{g} - \mathbf{\Omega} \times \Big(\mathbf{\Omega} \times \mathbf{r}_0\Big) = 0\,, \tag{3.3.12}$$

and for small deviations from equilibrium the motion of a long-enough pendulum is parallel to the Earth's surface to good approximation, with restoring force

$$\frac{\mathbf{F}}{m} + \mathbf{g} - \mathbf{\Omega} \times \Big(\mathbf{\Omega} \times \mathbf{r}\Big) \simeq -\omega^2(\mathbf{r} - \mathbf{r}_0)\,, \tag{3.3.13}$$

where standard arguments show why $\omega^2 \simeq g/L$ (see for instance the arguments leading to (2.5.9) for the bead on a rotating wire, in the limit where the wire's rotation goes to zero).

Defining $\mathbf{s} := \mathbf{r} - \mathbf{r}_0 \simeq x(t)\,\mathbf{e}_\phi + y(t)\,\mathbf{e}_\theta$, eq. (3.3.11) becomes

$$\frac{\partial^2 \mathbf{s}}{\partial t^2} \simeq -\omega^2 \mathbf{s} - 2\,\mathbf{\Omega} \times \frac{\partial \mathbf{s}}{\partial t}\,. \tag{3.3.14}$$

When evaluating $\mathbf{\Omega} \times \partial \mathbf{s}/\partial t$ we use

$$\mathbf{e}_z \times \mathbf{e}_r = \mathbf{e}_\phi \, \sin\theta\,, \quad \mathbf{e}_z \times \mathbf{e}_\theta = \mathbf{e}_\phi \, \cos\theta \quad \text{and} \quad \mathbf{e}_z \times \mathbf{e}_\phi = -(\mathbf{e}_r \, \sin\theta + \mathbf{e}_\theta \, \cos\theta) \tag{3.3.15}$$

and so when $\partial \mathbf{s}/\partial t = (\partial x/\partial t)\,\mathbf{e}_\phi + (\partial y/\partial t)\,\mathbf{e}_\theta$ we have

$$-2\,\mathbf{\Omega} \times \frac{\partial \mathbf{s}}{\partial t} = 2\Omega \left[\frac{\partial x}{\partial t}\Big(\sin\theta_0\,\mathbf{e}_r + \cos\theta_0\,\mathbf{e}_\theta\Big) - \frac{\partial y}{\partial t}\,\cos\theta_0\,\mathbf{e}_\phi \right]\,. \tag{3.3.16}$$

The vertical part of this just changes the value of the tension on the supporting pendulum string and so can be ignored for the present purposes.

The remaining two equations in the horizontal (*i.e.* $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$) directions are

$$\frac{\partial^2 x}{\partial t^2} \simeq -\omega^2 x - 2\,\Omega\cos\theta_0\,\frac{\partial y}{\partial t} \tag{3.3.17a}$$

$$\frac{\partial^2 y}{\partial t^2} \simeq -\omega^2 y + 2\,\Omega\cos\theta_0\,\frac{\partial x}{\partial t}\,. \tag{3.3.17b}$$

To solve this write $\mathfrak{z} = x + iy$ so that (3.3.17a)+i (3.3.17b) implies

$$\frac{\partial^2 \mathfrak{z}}{\partial t^2} \simeq -\omega^2 \mathfrak{z} + 2i\,\Omega\cos\theta_0\,\frac{\partial \mathfrak{z}}{\partial t}\,. \tag{3.3.18}$$

This has as solution $\mathfrak{z} = C_+\,e^{i\varpi_+ t} + C_-\,e^{i\varpi_- t}$ where $C_\pm$ are integration constants and both $\varpi_\pm$ satisfy $\varpi^2 - \omega^2 + 2i\varpi\,\Omega\cos\theta_0 = 0$ and so

$$\varpi_\pm = \Omega\cos\theta_0 \pm \sqrt{\omega^2 + \Omega^2\cos^2\theta_0} \simeq \pm\omega + \Omega\cos\theta_0 + \mathcal{O}(\Omega^2/\omega)\,, \tag{3.3.19}$$

where the approximate equality assumes $\omega \gg \Omega$ (as is the case for most pendula). Therefore $\mathfrak{z} = (C_+\,e^{i\omega t} + C_-\,e^{-i\omega t})e^{i\Omega t\cos\theta_0}$ and so choosing $t = 0$ to correspond to $x(0) = y(0) = 0$ implies $\mathfrak{z}(0) = C_+ + C_- = 0$, so $\mathfrak{z}(t) = Ce^{i\Omega t\cos\theta_0}\sin(\omega t)$, where C is real if the initial velocity is in the 'x' or $\mathbf{e}_\phi$ direction and is imaginary if the initial velocity is in the 'y' or $\mathbf{e}_\theta$ direction.

For real C we have

$$x(t) = 2C\cos(\Omega t\cos\theta_0)\sin(\omega t) \quad \text{and} \quad y(t) = 2C\sin(\Omega t\cos\theta_0)\sin(\omega t)\,. \tag{3.3.20}$$

When $\omega \gg \Omega$ this describes rapid oscillations with frequency ω along a line that is initially in the x or $\mathbf{e}_\phi$ (East-West) direction but slowly rotates with frequency $\Omega \cos\theta_0$: the famous Foucault rotation due to the Earth's rotation.

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3.3.3 Free fall near the Earth's surface

Consider next a projectile that moves under the influence only of gravity, and so in the frame rotating with the Earth satisfies (3.3.4) in the form

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} = \mathbf{g} - \boldsymbol{\Omega} \times \Big(\boldsymbol{\Omega} \times \mathbf{r}\Big) - 2\,\boldsymbol{\Omega} \times \frac{\partial \mathbf{r}}{\partial t}\,. \tag{3.3.21}$$

In the absence of fictitious forces this would have experienced only the acceleration of gravity and so move along the parabolic trajectory

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0(t-t_0) + \tfrac{1}{2}\,\mathbf{g}(t-t_0)^2 \qquad \text{(no fictitious forces)}\,. \tag{3.3.22}$$

How does this trajectory change due to the fictitious centrifugal and Coriolis forces?

A good approximation to the answer can be found if we recognize that the fictitious forces are small so the solution to (3.3.21) does not differ too much from (3.3.22). The means that we can to good approximation on the right-hand side replace $\mathbf{r}$ with $\mathbf{r}_0 \simeq R_\oplus\,\mathbf{e}_r$ in the centrifugal force and replace $\partial\mathbf{r}/\partial t$ with $\mathbf{v}_0 + \mathbf{g}(t-t_0) = \mathbf{v}_0 - g(t-t_0)\,\mathbf{e}_r$ in the Coriolis force. With these choices (3.3.21) becomes

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} \simeq \mathbf{g}_{\rm eff} - 2\,\boldsymbol{\Omega} \times \mathbf{v}_0 - 2\boldsymbol{\Omega} \times \mathbf{g}(t-t_0)\,, \tag{3.3.23}$$

where

$$\mathbf{g}_{\rm eff} = \mathbf{g} - \boldsymbol{\Omega} \times \Big(\boldsymbol{\Omega} \times \mathbf{r}_0\Big) = (-g + \Omega^2 R_\oplus \sin^2\theta_0)\,\mathbf{e}_r + \Omega^2 R_\oplus \sin\theta_0 \cos\theta_0\,\mathbf{e}_\theta\,, \tag{3.3.24}$$

and

$$-\boldsymbol{\Omega} \times \mathbf{g} = g\Omega\,\mathbf{e}_z \times \mathbf{e}_r = g\Omega\,\sin\theta_0\,\mathbf{e}_\phi\,. \tag{3.3.25}$$

Integrating (and assuming the motion changes the elevation negligibly relative to the Earth's radius and the latitude and longitude do not change appreciably) then gives

$$\frac{\partial \mathbf{r}}{\partial t} \simeq \mathbf{v}_0 + \Big[\mathbf{g}_{\rm eff} - 2\,\boldsymbol{\Omega} \times \mathbf{v}_0\Big](t-t_0) - \boldsymbol{\Omega} \times \mathbf{g}(t-t_0)^2 \tag{3.3.26}$$

and

$$\mathbf{r} \simeq \mathbf{r}_0 + \mathbf{v}_0(t-t_0) + \tfrac{1}{2}\Big[\mathbf{g}_{\rm eff} - 2\,\boldsymbol{\Omega} \times \mathbf{v}_0\Big](t-t_0)^2 - \tfrac{1}{3}\boldsymbol{\Omega} \times \mathbf{g}(t-t_0)^3\,. \tag{3.3.27}$$

Worked example: Particle dropped from rest

Consider for example the special case of a particle dropped from rest at a height $h \ll R_\oplus$ above the Earth's surface, so $\mathbf{v}_0 = 0$ and $\mathbf{r}_0 = (R_\oplus + h)\mathbf{e}_r$. In this case (3.3.26) and (3.3.27) become

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial t} &\simeq \mathbf{g}_{\text{eff}}(t-t_0) - \boldsymbol{\Omega}\times\mathbf{g}(t-t_0)^2 \\ &= (-g + \Omega^2 R_\oplus \sin^2\theta_0)(t-t_0)\,\mathbf{e}_r + \Omega^2 R_\oplus (t-t_0)\sin\theta_0\cos\theta_0\,\mathbf{e}_\theta \\ &\qquad + g\,\Omega(t-t_0)^2\,\sin\theta_0\,\mathbf{e}_\phi\end{aligned} \tag{3.3.28}$$

and

$$\begin{aligned}\mathbf{r} &\simeq \mathbf{r}_0 + \tfrac{1}{2}\mathbf{g}_{\text{eff}}(t-t_0)^2 - \tfrac{1}{3}\boldsymbol{\Omega}\times\mathbf{g}(t-t_0)^3 \\ &= \Big[R_\odot + h + \tfrac{1}{2}\Big(-g+\Omega^2 R_\oplus\sin^2\theta_0\Big)(t-t_0)^2\Big]\mathbf{e}_r + \tfrac{1}{2}\Omega^2 R_\oplus (t-t_0)^2\sin\theta_0\cos\theta_0\,\mathbf{e}_\theta \\ &\qquad + \tfrac{1}{3} g\,\Omega(t-t_0)^3\,\sin\theta_0\,\mathbf{e}_\phi\,.\end{aligned} \tag{3.3.29}$$

These show that an object dropped from rest does not land immediately below the dropping point. The centrifugal force also pushes the object in the North-South plane and the Coriolis force pushes it in the East-West plane since $\mathbf{e}_\theta$ points south and $\mathbf{e}_\phi$ points east. Given that the time taken to fall through the distance h is to a good approximation given by

$$t - t_0 = \sqrt{\frac{2h}{g - \Omega^2 R_\oplus \sin^2\theta_0}} \simeq \sqrt{\frac{2h}{g}} \tag{3.3.30}$$

the distance from the landing point to the point vertically under the drop point is

$$\frac{d_{NS}}{h} \simeq \frac{\Omega^2 R_\oplus}{g}\sin\theta_0\cos\theta_0 \quad \text{(south)} \qquad \text{and} \qquad \frac{d_{EW}}{h} \simeq \frac{2\Omega}{3}\sqrt{\frac{2h}{g}} \quad \text{(east)}. \tag{3.3.31}$$

Because $\cos\theta_0$ changes sign by passing through zero at the equator we see that centrifugal force pushes the object towards the south in the northern hemisphere and towards the north in the southern hemisphere. By contrast, the Coriolis force pushes the object towards the east in both hemispheres.

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When the initial velocity is not negligible $\mathbf{v}_0 = v_{r0}\,\mathbf{r} + v_{\theta 0}\,\mathbf{e}_\theta + v_{\phi 0}\,\mathbf{e}_\phi$ and so

$$-\boldsymbol{\Omega}\times\mathbf{v}_0 = \Omega\Big[v_{\phi 0}\,\sin\theta\,\mathbf{e}_r + v_{\phi 0}\,\cos\theta\,\mathbf{e}_\theta - (v_{r0}\,\sin\theta + v_{\theta 0}\,\cos\theta)\mathbf{e}_\phi\Big] \tag{3.3.32}$$

which shows that intial motion in the East-West direction gets deflected into the North-South-vertical plane while motion in the North-South and vertical directions get deflected into the East-West plane. In this case eqs. (3.3.26) and (3.3.27) become

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial t} &\simeq \mathbf{v}_0 + \Big[\mathbf{g}_{\text{eff}} - 2\,\boldsymbol{\Omega}\times\mathbf{v}_0\Big](t-t_0) - \boldsymbol{\Omega}\times\mathbf{g}(t-t_0)^2 \\ &= \Big[v_{r0} + \Big(-g + \Omega^2 R_\oplus\sin^2\theta_0 + 2v_{\phi 0}\Omega\sin\theta_0\Big)(t-t_0)\Big]\mathbf{e}_r \\ &\qquad + \Big[v_{\theta 0} + \Big(\Omega^2 R_\oplus \sin\theta_0\cos\theta_0 + 2v_{\phi 0}\Omega\cos\theta_0\Big)(t-t_0)\Big]\mathbf{e}_\theta \\ &\qquad\qquad + \Big[v_{\phi 0} - 2\Big(v_{r0}\,\sin\theta + v_{\theta 0}\,\cos\theta\Big)\Omega(t-t_0) + g\,\Omega(t-t_0)^2\,\sin\theta_0\Big]\mathbf{e}_\phi\end{aligned} \tag{3.3.33}$$

and

$$
\begin{aligned}
\mathbf{r} &\simeq \mathbf{r}_0 + \mathbf{v}_0(t-t_0) + \tfrac{1}{2}\Big[\mathbf{g}_{\rm eff} - 2\,\boldsymbol{\Omega}\times\mathbf{v}_0\Big](t-t_0)^2 - \tfrac{1}{3}\boldsymbol{\Omega}\times\mathbf{g}(t-t_0)^3 \\
&= \Big[R_\odot + h + v_{r_0}(t-t_0) + \tfrac{1}{2}\Big(-g + \Omega^2 R_\oplus \sin^2\theta_0 + 2v_{\phi 0}\Omega\sin\theta_0\Big)(t-t_0)^2\Big]\mathbf{e}_r \\
&\qquad + \Big[v_{\theta_0}(t-t_0) + \tfrac{1}{2}\Big(\Omega^2 R_\oplus \sin\theta_0\cos\theta_0 + 2v_{\phi 0}\Omega\cos\theta_0\Big)(t-t_0)^2\Big]\mathbf{e}_\theta \\
&\qquad\qquad + \Big[v_{\phi 0}(t-t_0) - \Big(v_{r0}\,\sin\theta + v_{\theta 0}\,\cos\theta\Big)\Omega(t-t_0)^2 + \tfrac{1}{3}g\,\Omega(t-t_0)^3\,\sin\theta_0\Big]\mathbf{e}_\phi\,.
\end{aligned}
\tag{3.3.34}
$$

Worked example: Sharpshooter's aim

For instance suppose a sharpshooter fires a high-velocity bullet ($v_0 \gtrsim 1000$ m/s) that is initially horizontal (so $v_{r0} = 0$). In what direction must the shot be taken in order to hit a target a distance d away that is due East of the shooter? In this case setting $v_{r0} = 0$ implies the components of (3.3.34) become

$$r(t) \simeq R_\odot + h + \tfrac{1}{2}\Big(-g + \Omega^2 R_\oplus \sin^2\theta_0 + 2v_{\phi 0}\Omega\sin\theta_0\Big)(t-t_0)^2 \tag{3.3.35a}$$

$$s_\theta(t) \simeq v_{\theta_0}(t-t_0) + \tfrac{1}{2}\Big(\Omega^2 R_\oplus \sin\theta_0\cos\theta_0 + 2v_{\phi 0}\Omega\cos\theta_0\Big)(t-t_0)^2 \tag{3.3.35b}$$

$$s_\phi(t) \simeq v_{\phi 0}(t-t_0) - v_{\theta 0}\,\cos\theta\Omega(t-t_0)^2 + \tfrac{1}{3}g\,\Omega(t-t_0)^3\,\sin\theta_0\,. \tag{3.3.35c}$$

Since $\Omega R_\oplus \simeq 500$ m/s a high-velocity bullet travels fast enough that Coriolis effects are small relative to the initial velocity for the entire length of the bullet's free fall. As a result the time taken to travel the distance $s_\phi(T) = d$ is approximately $T = t - t_0 \simeq d/v_{\phi 0}$. To hit the target the North-South component $v_{\theta 0}$ of the initial velocity must be chosen so that $s_\theta(T) = 0$, which requires

$$v_{\theta 0} \simeq -\tfrac{1}{2}\Big(\Omega R_\oplus \sin\theta_0 + 2v_{\phi 0}\Big)\Omega T\cos\theta_0 \tag{3.3.36}$$

and so the shot must be an angle α south of east where

$$\tan\alpha = \frac{v_{\theta 0}}{v_{\phi 0}} \simeq -\left(1 + \frac{\Omega R_\oplus \sin\theta_0}{2v_{\phi 0}}\right)\frac{\Omega d}{v_{\phi 0}}\cos\theta_0 \simeq \frac{\Omega d}{v_{\phi 0}}\cos\theta_0 \tag{3.3.37}$$

Notice that because $\cos\theta_0 < 0$ in the southern hemisphere the aiming point there must be slightly north of the target.

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3.4 Orbital Lagrange Points

As a last application of rotating reference frames let us consider the motion of a three bodies mutually interacting through Newton's Law of universal gravitation. In this case the full problem does not have simple solutions – indeed it was the study of this problem that led to the discovery of chaos and chaotic evolution (more about which in §8) – so numerical methods are usually applied.

A situation for which analytic progess is possible is the situation where the third body is much less massive than the other two. In this case the larger two bodies to a first approximation move along the two-body orbits discussed in §1.2.2 and the third body moves within the time-dependent potential provided by the other two bodies. We explore this motion here in the special case that the larger two bodies move in a circular orbit and the third body also moves in the same plane (a not unusual situations, say, if the bodies are the Sun, a planet and an asteroid). We choose to number the bodies in order of decreasing mass: $m_1 > m_2 \gg m_3$.

In this kind of situation it is useful to adopt coordinates that rotate with the same angular frequency as the orbit of the two larger bodies, which for a circular orbit of radius a is given by (*c.f.* eq. (1.2.41))

$$\Omega = \frac{2\pi}{T} = \sqrt{\frac{GM}{a^3}} \,. \tag{3.4.1}$$

To this end we adopt coordinates with origin at the centre of mass of the two more massive bodies, and with a z-axis perpendicular to the orbital plane so all three particles satisfy $z = 0$ for all time. Because the reference frame rotates with the orbital frequency Ω we can choose the x-axis to point always at the two heavy bodies. The distance from the centre of mass to each of the two massive objects is

$$x_1 = \frac{\mu a}{m_1} = \frac{m_2 a}{m_1 + m_2} \text{and} \quad x_2 = \frac{\mu a}{m_2} = \frac{m_1 a}{m_1 + m_2} \,, \tag{3.4.2}$$

for all time. The positive x-axis points to the second object (the planet) and the negative axis points to the largest object (the Sun).

We denote the coordinates of the third body in the orbital plane by (x, y, z), but choose the x- and y-axes to rotate about the z-axis with angular speed Ω, so $\tilde{z} = z$ and

$$\begin{aligned} \tilde{x} &= x \cos(\Omega t) + y \sin(\Omega t) \\ \tilde{y} &= -x \sin(\Omega t) + y \cos(\Omega t) \,. \end{aligned} \tag{3.4.3}$$

The kinetic energy for motion of particle 3 in the rotating frame then becomes

$$K = \tfrac{1}{2} m_3 \Big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \Big) = \tfrac{1}{2} m_3 \Big[(\dot{\tilde{x}} - \Omega \tilde{y})^2 + (\dot{\tilde{y}} + \Omega \tilde{x})^2 + \dot{\tilde{z}}^2 \Big] = \tfrac{1}{2} m_3 \Big(\dot{\tilde{\mathbf{r}}} + \mathbf{\Omega} \times \tilde{\mathbf{r}} \Big)^2 \,. \tag{3.4.4}$$

and so

$$\frac{\partial K}{\partial \dot{\tilde{\mathbf{r}}}} = m_3 \Big(\dot{\tilde{\mathbf{r}}} + \mathbf{\Omega} \times \tilde{\mathbf{r}} \Big) \quad \text{and} \quad \frac{\partial K}{\partial \tilde{\mathbf{r}}} = m_3 \Big[\dot{\tilde{\mathbf{r}}} \times \mathbf{\Omega} - \mathbf{\Omega} \times \Big(\mathbf{\Omega} \times \tilde{\mathbf{r}} \Big) \Big] \,. \tag{3.4.5}$$

Varying the Lagrangian $L = K - V$ then gives the equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{\tilde{\mathbf{r}}}} \right) - \frac{\partial L}{\partial \tilde{\mathbf{r}}} = m_3 \Big(\ddot{\tilde{\mathbf{r}}} + \mathbf{\Omega} \times \dot{\tilde{\mathbf{r}}} \Big) - m_3 \Big[\dot{\tilde{\mathbf{r}}} \times \mathbf{\Omega} - \mathbf{\Omega} \times \Big(\mathbf{\Omega} \times \tilde{\mathbf{r}} \Big) \Big] + \frac{\partial V}{\partial \tilde{\mathbf{r}}} = 0 \tag{3.4.6}$$

in agreement with (3.3.4) (with $\mathbf{g} = 0$) where the applied force is $\mathbf{F} = -\partial V / \partial \tilde{\mathbf{r}}$.

In the present instance the potential energy of the light particle due to its attraction to the heavier particles is

$$V = -\frac{G m_1 m_3}{r_{13}} - \frac{G m_2 m_3}{r_{23}} \,, \tag{3.4.7}$$

where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ is the distance between particle i and particle j. For motion in the x-y plane the distances appearing in the potential become

$$r_{13}^2 = \left[\tilde{x} + \left(\frac{m_2}{m_1+m_2}\right) a\right]^2 + \tilde{y}^2 \quad \text{and} \quad r_{23}^2 = \left[\tilde{x} - \left(\frac{m_1}{m_1+m_2}\right) a\right]^2 + \tilde{y}^2 \,, \tag{3.4.8}$$

and so the equations of motion for the coordinates $(\tilde{x}, \tilde{y})$ of the third object in the orbital plane reduce to

$$\begin{aligned} m_3 \ddot{\tilde{x}} &= 2m_3\Omega\dot{\tilde{y}} + m_3\Omega^2\tilde{x} - \frac{\partial V}{\partial \tilde{x}} \\ m_3 \ddot{\tilde{y}} &= -2m_3\Omega\dot{\tilde{x}} + m_3\Omega^2\tilde{y} - \frac{\partial V}{\partial \tilde{y}} \,. \end{aligned} \tag{3.4.9}$$

We seek static solutions to these equations. Such solutions are called *Lagrange points* and are points where the gravitational and centrifugal forces all precisely balance so that a particle that starts there at rest remains in the same position relative to the two massive bodies, with all three bodies rotating together with a common angular frequency Ω.

The above equations in this case reduce to

$$\Omega^2\tilde{x} = \frac{1}{m_3}\frac{\partial V}{\partial \tilde{x}} = \frac{Gm_1}{r_{13}^3}\left[\tilde{x} + \left(\frac{m_2}{m_1+m_2}\right) a\right] + \frac{Gm_2}{r_{23}^3}\left[\tilde{x} - \left(\frac{m_1}{m_1+m_2}\right) a\right] \tag{3.4.10a}$$

$$\Omega^2\tilde{y} = \frac{1}{m_3}\frac{\partial V}{\partial y} = \left(\frac{Gm_1}{r_{13}^3} + \frac{Gm_2}{r_{23}^3}\right)\tilde{y} \,. \tag{3.4.10b}$$

The simplest examples of Lagrange points lie along the same axis that connects the two massive objects (for which $y = 0$). In this case the second equation is trivially satisfied while the first equation reduces to

$$\Omega^2\tilde{x} = Gm_1 \frac{\tilde{x} + \left(\frac{m_2}{m_1+m_2}\right) a}{\left|\tilde{x} + \left(\frac{m_2}{m_1+m_2}\right) a\right|^3} + Gm_2 \frac{\tilde{x} - \left(\frac{m_1}{m_1+m_2}\right) a}{\left|\tilde{x} - \left(\frac{m_1}{m_1+m_2}\right) a\right|^3} \,. \tag{3.4.11}$$

This equation has three solutions for x, as is most easily seen graphically by showing how the right-hand side has three branches that are each crossed once by the straight line that represnts the left-hand side (see Fig. 11). There is one solution each in the regimes: to the left of both massive objects; to the right of both massive objects; in between the two massive objects.

In the Earth-Sun system the solution in between the Sun and the Earth is conventionally called $L1$; the solution further from the Sun than the Earth is called $L2$ and the solution on the far side of the Sun from us is called $L3$. $L1$ is a useful place to place satellites that monitor the Sun since (unlike for orbits that circle the Earth) the Sun is never obscured by the Earth. $L2$ is a useful place for placing satellites that observe the rest of the Universe because there the Earth always obscures the Sun (keeping it from interfering with measurements). Because

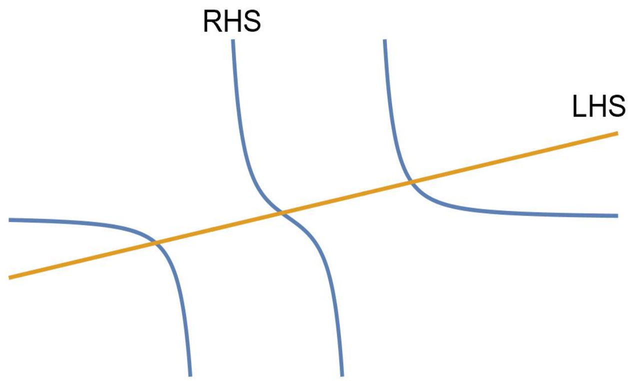

Figure 11. Plot of the left-hand side (LHS) and right-hand side (RHS) of eq. (3.4.11) vs $\tilde{x}$, showing why it has three solutions.

$L3$ is always hidden from us by the Sun it would be a perfect place to put something you'd like people on Earth never to see.

There are also solutions that do not lie along the line connecting the two massive sources. In this case dropping the common factor $\tilde{y}$ from (3.4.10b) implies

$$\Omega^2 = \frac{Gm_1}{r_{13}^3} + \frac{Gm_2}{r_{23}^3} \tag{3.4.12}$$

and using this in (3.4.10a) to eliminate Ω^2 then implies

$$\frac{Gm_1m_2}{r_{13}^3} = \frac{Gm_1m_2}{r_{23}^3} \tag{3.4.13}$$

and so $r_{13} = r_{23}$. But using this in (3.4.12) then implies

$$\Omega^2 = \frac{G(m_1+m_2)}{r_{13}^3} = \frac{G(m_1+m_2)}{r_{23}^3} \tag{3.4.14}$$

and comparing this with Kepler's third law $\Omega^2 = G(m_1+m_2)/a^3$ – see eq. (1.2.42) – then shows $r_{13} = r_{23} = a$ so the three bodies must lie at the corners of an equilateral triangle. That means the third body lies at the same radius from body 1 as does body 2, but leads it or follows it in this orbit by 60^o.

These two solutions are conventionally called $L4$ and $L5$. Although a use for these has not yet been found for artificial satellites in the Earth-Sun system, asteroids tend to accumulate at $L4$ and $L5$ for the Jupiter-Sun system; a class of asteroids known as *Trojan asteroids*. These seem to be roughly as numerous as are the asteroids in the regular asteroid belt, and the identification of these regions as static points helps understand why this is so.

4 Rigid Bodies

To this point we have discussed systems involving many atoms in a general way, mostly celebrating how Newton's Laws are recursive in the sense that they take the same form when written in terms of a system's microscopic constituents (such as its underlying atoms) or in terms of macroscopic subsets of constituents (like billiard balls colliding or the Sun moving around the Earth). This section – and §5 to follow – now ask in more detail what can be said about the motion of specific macroscopic objects individually.

We here start the process with the discussion of *rigid bodies*: bodies defined by the condition that all of their constituent atoms have fixed relative distances, as expressed by the holonomic constraint

$$r_{ab} := |\mathbf{r}_a - \mathbf{r}_b| = c_{ab} \qquad \text{for all pairs } (a, b)\,, \tag{4.0.1}$$

where for each pair the quantity c_{ab} is a fixed number. Intuitively this seems like it should be a good description of many everyday objects, everything from a rock to any other type hard place. But it is also clearly not a good description of all the bodies we see around us, such as cats, drops of water and other more malleable objects whose atoms move more freely relative to one another (more about *e.g.* fluids in §9). So it is worth starting by stepping back and asking why rigid bodies are interesting enough to deserve such detailed attention.

Why Rigid Bodies?

A useful way to organize how to think about matter in bulk (and physics more generally, as it happens) is to think about the relative scales that appear in any physical system. Nature famously serves up for study objects with a wide variety of sizes and shapes, running from subnuclear particles on the smallest of length scales up to clusters of galaxies on the largest. At any given time we are limited in the resolution of what we can see; for much of history we were as ignorant about smaller objects (like microbes or atoms or nuclei) as we now are about things that are smaller still (current measurements cannot resolve objects smaller than 10^{-20} m).

But even if we can measure the size of something small we might not care what the internal degrees of freedom are doing if our only concern is with the overall motion of an object's centre of mass. What is meant by 'small' here is of course relative, since it might be the size of the Earth that is regarded as small when describing its orbit about the Sun, or our entire galaxy might be small when asking about the average motion of matter within the observed universe. The process of ignoring internal degrees of freedom when thinking big is sometimes called 'coarse graining'.

For the present purposes what is important is that there are two different reasons why it might be a good approximation to ignore the internal degrees of freedom of a macroscopic object. One is that the object's linear size (*e.g.* its radius) is smaller than the minimum length scales of interest, such is the case for the Earth when describing its orbit. In this case an error comparable to the Earth's radius $R_\oplus \simeq 6.4 \times 10^3$ km only matters for an orbit of radius 1.5×10^8 km if we want to know orbital properties with a precision better than 0.004%. In this regime the object's centre of mass contains all of the useful information about its motion.

But a logically independent reason to be able to ignore internal degrees of freedom arises when the size of an object is *not* negligible but the transfer of energy into internal degrees of freedom is small enough to be ignored.[9] This situation can arise if the energy associated with the motion of the centre of mass is much smaller than the energy needed to make a significant number of constituents move relative to one another, such as for the collision of two balls on a pool table. From this perspective it is the nature of the interatomic forces that make the motion of a billiard ball more resemble a rigid body than does the motion of an equal-sized drop of water.

Rigid bodies describe the ideal in which internal forces are so strong that internal degrees of freedom experience no relative motion at all. They are relevant in practice because this is a good approximation for the very common situation where energy transfer to internal degrees of freedom is negligible. And they are worth detailed study because their motion differs in interesting ways from simple centre-of-mass movement (as anyone knows who has seen the motion of a spinning ball or a gyroscope).

4.1 Kinematics of Rigid Body Motion

The first step is to describe rigid-body motion – 'kinematics' – and the next step is to ask what Newton's Laws say about how this motion evolves given a set of applied forces – 'dynamics'. We start here with kinematics. In particular, how many generalized coordinates are needed to describe rigid-body motion?

For a body consisting of N atoms there are $3N$ degrees of freedom to which eqs. (4.0.1) impose a total of $\frac{1}{2}N(N-1)$ constraints. The net number of degrees of freedom cannot simply be the difference of these two numbers because for large N there are many more constraints than there are total degrees of freedom. But clearly not all of the constraints are independent because the position of any particular atom is uniquely specified once its relative displacement is known from any 3 noncollinear atoms.

So the positions of all atoms can be inferred from the constraints once one knows the positions $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ of just three noncollinear atoms. But even the 9 coordinates needed to specify these three atoms are not independent because they remain subject to the constraints $r_{12} = c_{12}$, $r_{13} = c_{13}$ and $r_{23} = c_{23}$, suggesting that $9 - 3 = 6$ generalized coordinates should suffice to describe rigid-body motion (regardless of how large N is).

[9] It can sometimes happen that the detailed motion of internal degrees of freedom can still be ignored even when they absorb significant energy, such as when this transfer can be parameterized in terms of an 'effective' description, like a frictional force or a heat capacity.

The need for six coordinates can also be seen more constructively: three coordinates are needed to specify the position, $\mathbf{r}_1$, of the first atom. Then two more coordinates are needed to specify the position of the second atom because it must be located somewhere on the surface of a sphere of radius r_{12}. The third atom's location then takes only one more coordinate because the constraints force it to reside on the intersection of two spheres, one of radius r_{13} centred on particle 1 and another of radius r_{23} centred on particle 2. This means that particle 3 lies on a circle of revolution obtained by rotating about the axis defined by the positions of particles 1 and 2, with known radius.

Three of the six degrees of freedom required to describe rigid body motion are the three components of the object's centre of mass (as defined in (1.3.4) and (1.3.5) and repeated again here for ease of reference)

$$\mathbf{R} := \frac{1}{M}\sum_a m_a \mathbf{r}_a \qquad \text{where} \qquad M := \sum_a m_a\,. \tag{4.1.1}$$

We can imagine this to be the location of the first of the three noncollinear reference atoms described above (relative to which the constraints (4.0.1) uniquely determine the positions of all other atoms). To this we must add three rotation angles (two describing the direction of particle 2 from particle 1 and the third describing the direction to particle 3 from the axis connecting particles 1 and 2).

The need for three spatial position variables and three rotation angles also makes sense from another point of view. We have seen that spacetime symmetries – translations in space and time, rotations and Galilean transformations – play an important role in classical mechanics, with each symmetry corresponding to a conservation law. In particular conservation of linear and angular momentum seem to indicate that nature's laws are invariant under spatial translations and rotations.

From this point of view rigid bodies are very generally described by a centre-of-mass coordinate because we can generate many solutions to the equations of motion just by translating any particular solution, provided that this solution is not itself translation invariant. Any configuration obtained by translating a particular solution must also be a solution because the equations being solved are assumed to be translation invariant. For example the expression for the orbit for two particles orbiting one another fixes *e.g.* the orbital radius but leaves the centre of mass of the orbiting pair completely unspecified, and it is this centre of mass that changes when the original solution is translated. The solutions to translation-invariant equations cannot be unique: they must contain a separate solution for each possible translation and this is why centre of mass position is always one of the coordinates needed to describe the behaviour of any finite-sized objects (including rigid ones).

Precisely the same reasoning applies for rotations if the solution to rotation-invariant equations are not themselves rotation invariant, and this is ultimately why finite sized bodies (including rigid ones) also involve three angular coordinates. What is unique about *rigid* bodies is that the six coordinates implied by translation and rotation invariance are the only ones required to describe the motion. Part of the utility of rigid body motion is that the degrees of freedom involved (centre of mass position and angular orientation) are actually *generic* to any finite-sized object.

4.1.1 Instantaneous Angular Velocity

The next step is to characterize the angular variables as precisely as eq. (4.1.1) does for the centre of mass variables. Assuming the first reference atom is placed at the centre of mass, the other two can be taken to lie along two perpendicular directions seen from the centre of mass, and so define two orthogonal unit vectors $\mathbf{e}_1'$ and $\mathbf{e}_2'$ pointing in these two directions. The third vector required to form a left-handed orthonormal basis is then defined by $\mathbf{e}_3' = \mathbf{e}_1' \times \mathbf{e}_2'$. In this case any rotation of the rigid body about its centre of mass is described by an active rotation, $R_{ij}\, \mathbf{e}_j'$, of the basis $\{\mathbf{e}_i'\} = \{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ relative to a fixed reference orthonormal basis $\{\mathbf{e}_i\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ defined in an inertial reference frame instantaneously at rest relative to the centre of mass. This makes the rotations describing rigid-body rotations take the general form described in §2.3.4.

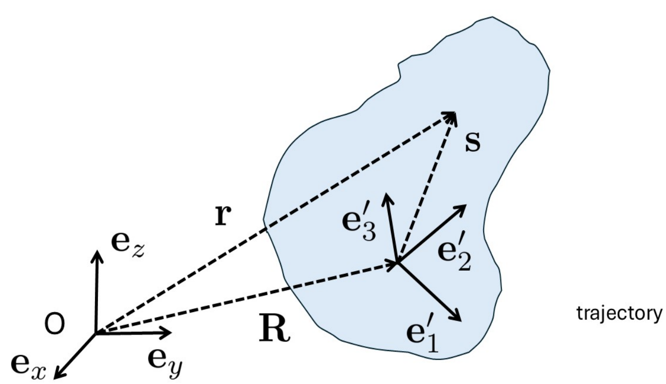

Figure 12. Illustration of the coordinate systems used when describing rigid body motion. The basis $\mathbf{e}_i$ is inertial and the basis $\mathbf{e}_i'$ is attached to the body's centre of mass and moves with it.

To make things even more concrete it is useful to specialize temporarily to infinitesimal rotations. Consider therefore a rigid body with centre of mass position $\mathbf{R}$. Denote the position of a point on the body relative to the body's centre of mass by $\mathbf{s}$ and denote the position of this same point relative to an inertial observer at O by $\mathbf{r}$ (see Fig. 12), so $\mathbf{r} = \mathbf{R} + \mathbf{s}$. From the above arguments a small change d$\mathbf{r}$ can be regarded as a small displacement d$\mathbf{R}$ of the body's centre of mass plus a small rotation $\mathrm{d}\mathbf{s} = \mathrm{d}\boldsymbol{\phi} \times \mathbf{s}$ about the centre of mass:

$$\mathrm{d}\mathbf{r} = \mathrm{d}\mathbf{R} + \mathrm{d}\boldsymbol{\phi} \times \mathbf{s}\,, \tag{4.1.2}$$

where $\delta\boldsymbol{\phi} = \mathbf{n}\,\mathrm{d}\phi$ points in the direction of the instantaneous axis of rotation[10] and has magnitude equal to the size of the rotation about this axis (compare with eq. (2.3.22)).

Dividing (4.1.2) by the time $\mathrm{d}t$ taken to move through this displacement shows that the velocity of the point $\mathbf{v}$, the velocity of the centre of mass, $\mathbf{V}$, and the angular velocity $\boldsymbol{\Omega}$, whose definitions are

$$\mathbf{v} := \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,, \qquad \mathbf{V} := \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}\,, \qquad \boldsymbol{\Omega} := \frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t}\,, \tag{4.1.3}$$

must be related by

$$\mathbf{v} = \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{s}\,. \tag{4.1.4}$$

This shows how the velocity of any particular point on a rigid body can be expressed in terms of the position $\mathbf{s}$ of the point relative to the centre of mass and two global velocities $\mathbf{V}$ and $\boldsymbol{\Omega}$ that describe the motion of the body as a whole. Consequently the quantities $\mathbf{V}$ and $\boldsymbol{\Omega}$ provide the six velocity components needed to describe rigid-body motion. Notice that $\boldsymbol{\Omega}$ is *not* assumed to be time-independent *a priori* (unlike in §3.3) and §4.3 below is devoted to deriving its equations of motion.

4.1.2 Moments of Inertia

Knowing the connection between the velocity $\mathbf{v}$ of a specific point on a rigid body and the generalized velocities for rigid-body motion allows the kinetic energy to be expressed directly in terms of $\mathbf{V}$ and $\boldsymbol{\Omega}$. Eq. (1.4.3) (repeated here for convenience using the notation of this section) states

$$K = \sum_a \tfrac{1}{2}\, m_a\, \mathbf{v}_a^2 = \tfrac{1}{2}\, M\, \mathbf{V}^2 + E_{\rm rot} \tag{4.1.5}$$

where (as in Fig. 12) $\mathbf{r}_a = \mathbf{R} + \mathbf{s}_a$ and we separate off the internal energy, $E_{\rm int}$, associated with the time derivative of $\mathbf{s}_a$, but this time label it $E_{\rm rot}$ to emphasize that for rigid bodies the only internal motion possible relative to the centre of mass is a rotation:

$$E_{\rm rot} := \sum_a \tfrac{1}{2}\, m_a\, \dot{\mathbf{s}}_a^2 = \sum_a \tfrac{1}{2}\, m_a \Big(\boldsymbol{\Omega} \times \mathbf{s}_a\Big)^2\,, \tag{4.1.6}$$

where the last equality uses (4.1.4), in the form $\dot{\mathbf{s}}_a = \boldsymbol{\Omega} \times \mathbf{s}_a$.

Simplifying using the vector identity (A.3.15) allows the internal energy to be rewritten

$$E_{\rm rot} := \sum_a \tfrac{1}{2}\, m_a \Big(\boldsymbol{\Omega} \times \mathbf{s}_a\Big)^2 = \sum_a \tfrac{1}{2}\, m_a \Big[\Omega^2 \mathbf{s}_a^2 - (\boldsymbol{\Omega} \cdot \mathbf{s}_a)^2\Big] = \tfrac{1}{2}\, I_{ij}\, \Omega_i\, \Omega_j \tag{4.1.7}$$

where the last equality factorizes out the a-independent components Ω_i of $\boldsymbol{\Omega}$ and lumps the sum over a into the symmetric 3×3 coefficient matrix I_{ij}:

$$I_{ij} := \sum_a m_a \Big(\delta_{ij}\, \mathbf{s}_a^2 - s_{ai}\, s_{aj}\Big)\,. \tag{4.1.8}$$

[10]Notice that it is only infinitesimal rotations, $R_{ij} = \delta_{ij} + \Theta_{ij}$ in the notation of §2.3.4, that define an instantaneous axis of rotation through $\Theta_{ij} = \epsilon_{ijk} n_k\, \mathrm{d}\phi$. There is no similar uniquely defined vector for a finite 3D rotation R_{ij} that is not close to the identity matrix.

Eq. (4.1.8) defines the *moment of inertia tensor* – sometimes just 'inertia tensor' – which can be written out explicitly in matrix form (not to be confused with the unit matrix I) as

$$\mathbb{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} = \sum_a m_a \begin{pmatrix} y_a^2 + z_a^2 & -x_a y_a & -x_a z_a \\ -x_a y_a & x_a^2 + z_a^2 & -y_a z_a \\ -x_a z_a & -y_a z_a & x_a^2 + y_a^2 \end{pmatrix}, \tag{4.1.9}$$

which denotes the components of $\mathbf{s}_a$ by $\mathbf{s}_a = x_a\,\mathbf{e}_x + y_a\,\mathbf{e}_y + z_a\,\mathbf{e}_z$.

In terms of the matrix $\mathbb{I}$ the internal energy (4.1.7) can also be written

$$E_{\rm int} = \frac{1}{2}\boldsymbol{\Omega}^T \mathbb{I}\,\boldsymbol{\Omega}\,, \tag{4.1.10}$$

and this shows how to make contact with the notion of moment of inertia encountered in elementary mechanics classes. In these a body's moment of inertia is defined relative to a specific rotation axis. If an object is rotated with angular velocity Ω about an axis passing through the centre of mass in the direction defined by a unit vector $\mathbf{n}$ then the energy of rotation is written $E_{\rm rot} = \frac{1}{2}\mathfrak{I}\,\Omega^2$ where $\mathfrak{I}$ is the moment of inertia about this axis. Comparing this with (4.1.10) using $\boldsymbol{\Omega} = \Omega\mathbf{n}$ shows that $\mathfrak{I}$ is related to I_{ij} by

$$\mathfrak{I} = I_{ij} n_i n_j = \mathbf{n}^T\,\mathbb{I}\,\mathbf{n}\,. \tag{4.1.11}$$

Expression (4.1.9) shows that the moment of inertia is additive, much as is the total mass $M = \sum_a m_a$. This is useful, particularly when one does not know explicitly where each individual atom is and instead only knows their mass density $\rho(\mathbf{x})$, defined by

$$\rho(\mathbf{x}, t)\,{\rm d}V = \rho(\mathbf{x}, t)\,{\rm d}^3x = \sum_{a\in {\rm d}V} m_a \tag{4.1.12}$$

where ${\rm d}V = {\rm d}^3x = {\rm d}x\,{\rm d}y\,{\rm d}z$ is the volume of a very small parallelipiped centred on the point $\mathbf{x}$, whose sides in the three spatial directions have length ${\rm d}x$, ${\rm d}y$ and ${\rm d}z$, respectively. It is imagined here that these sizes, ${\rm d}x_i$, are infinitesimally small compared to the spatial resolution available for a particular application but very large compared to the typical spacing between atoms. This definition ensures that integrating $\rho(\mathbf{x}, t)$ over the volume of a macroscopic object X gives its mass:

$$\int_X {\rm d}^3x\;\rho = \sum_{a\in X} m_a = M\,. \tag{4.1.13}$$

Since ${\rm d}x_i$ is at the limits of the spatial resolution available the sum appearing in (4.1.9) can be evaluated as an integral where all of the atoms within the same parallelipiped are taken to share the same position, so

$$\rho(\mathbf{x})\,x_i\,x_j\,{\rm d}V = \sum_{a\in{\rm d}V} m_a x_{ia}\,x_{ja} \tag{4.1.14}$$

and therefore the inertia tensor of a macroscopic body X can be written in a much more useful integral form

$$I_{ij} = \int_X \mathrm{d}^3x \; \rho(\mathbf{x}) \left(\delta_{ij}\, \mathbf{x}^2 - x_i\, x_j\right), \tag{4.1.15}$$

where $\mathbf{x}$ is the position of a point in body X relative to its centre-of-mass position.

In general the tensor $\mathbb{I}$ is not diagonal – as (4.1.9) also shows – though the values taken by each of the components I_{ij} (and in particular the off-diagonal ones) are frame-dependent inasmuch as they take different values when one rotates to a new set of basis vectors $\mathbf{e}_i \to \widetilde{\mathbf{e}}_i = R_{ij}\mathbf{e}_j$. We saw in §2.3.4 that under such a (passive) rotation the components of any vector – and in particular the vectors $\boldsymbol{\Omega}$ and $\mathbf{s}_a$ – transform to $\widetilde{\Omega}_i = R_{ij}\Omega_j$ and $\tilde{x}_{ia} = R_{ij}\, x_{ja}$. This implies the matrix elements I_{ij} after such a rotation differ from those before rotating by a similarity transformation:

$$\tilde{I}_{ij} = R_{ik}R_{jl}I_{kl}\,, \qquad \text{or, in matrix form,} \qquad \tilde{\mathbb{I}} = R\,\mathbb{I}\,R^T\,. \tag{4.1.16}$$

Like any real symmetric matrix there always exists an orthogonal similarity transformation, S, satisfying $S^TS = SS^T = I$, for which

$$S\,\mathbb{I}\,S^T = \begin{pmatrix} \mathcal{I}_1 & 0 & 0 \\ 0 & \mathcal{I}_2 & 0 \\ 0 & 0 & \mathcal{I}_3 \end{pmatrix} \quad \text{is diagonal.} \tag{4.1.17}$$

Then eigenvectors of the matrix $\mathbb{I}$ and are called the *principal axes of inertia* and because they are obtained by a rotation S they are orthogonal. The basis vectors obtained by this particular rotation, $\tilde{\mathbf{e}}_i = S_{ij}\mathbf{e}_j$, provide a privileged reference frame in terms of which the inertia tensor is diagonal and so is particular simple. This frame moves with the rigid body and has an orientation adapted to the body's shape, and so is called the *body frame*.

The diagonal elements (or eigenvalues) of the matrix $\mathbb{I}$ are called the *principal moments of inertia*. Unlike the individual components I_{ij} in some random reference frame, the principal moments are invariant under rotations and so can be regarded as intrinsic properties of the rigid body. Physical properties are often simplest when expressed in terms of the principal axes. For instance the rotational kinetic energy is a particularly simple function of the components Ω_1, Ω_2 and Ω_3 of $\boldsymbol{\Omega}$ in this frame since (4.1.17) allows (4.1.7) to be written

$$E_{\rm rot} = \tfrac{1}{2}\Big(\mathcal{I}_1\,\Omega_1^2 + \mathcal{I}_2\,\Omega_2^2 + \mathcal{I}_3\,\Omega_3^2\Big)\,. \tag{4.1.18}$$

The definitions of the principal moments imply that no principal moment can be larger than the sum of the other two principal moments:

$$\mathcal{I}_1 \le \mathcal{I}_2 + \mathcal{I}_3\,, \quad \mathcal{I}_2 \le \mathcal{I}_1 + \mathcal{I}_3 \quad \text{and} \quad \mathcal{I}_3 \le \mathcal{I}_1 + \mathcal{I}_2\,. \tag{4.1.19}$$

This is most easily seen by specializing (4.1.8) to coordinates that are aligned with the principal axes of inertia. In this case the agreement of (4.1.9) with the diagonal form (4.1.17) shows that

$$\mathcal{I}_1 = \sum_a m_a \Big(\hat{x}_{2a}^2 + \hat{x}_{3a}^2\Big) \quad \mathcal{I}_2 = \sum_a m_a \Big(\hat{x}_{1a}^2 + \hat{x}_{3a}^2\Big) \quad \text{and} \quad \mathcal{I}_3 = \sum_a m_a \Big(\hat{x}_{1a}^2 + \hat{x}_{2a}^2\Big) \quad (4.1.20)$$

(where the hats on $\hat{x}_{ia}$ are meant as reminders that these expressions only holds when the coordinates are adapted to the principal axes of inertia). In particular, eqs. (4.1.20) imply $\mathcal{I}_1 + \mathcal{I}_2 = \sum_a m_a(\hat{x}_{1a}^2 + \hat{x}_{2a}^2 + 2\hat{x}_{3a}^2) \geq \mathcal{I}_3 = \sum_a m_a(\hat{x}_{1a}^2 + \hat{x}_{2a}^2)$ (and similarly for the other pairs of sums of $\mathcal{I}_i$'s).

The more asymmetrical an object is the more the principal moments differ from one another, motivating the definitions:

- *Spherical top:* a rigid body with three equal principal moments: $\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}_3$. This is the most symmetric situation because in this case the inertia tensor is proportional to the unit matrix and so is diagonal in *any* reference frame. Any choice of orthogonal basis vectors is in this case forms a principal axis.

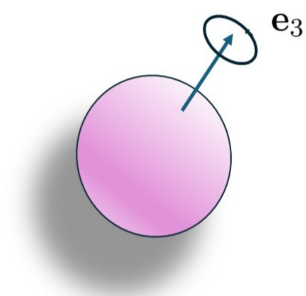

Figure 13. Example of a spherical top.

- *Symmetrical top:* a rigid body for which $\mathcal{I}_1 = \mathcal{I}_2 \neq \mathcal{I}_3$. In this case the choice of principal axes is again not unique because an arbitrary rotation in the 1-2 plane does not change the diagonal form of I_{ij}. This is what would be expected for an object with an axis of rotational symmetry, though complete cylindrical symmetry is not required because invariance under any discrete rotations except 180^o about a fixed axis suffice to imply $I_1 = I_2$.

- *Asymmetric top:* a rigid body where all three principle moments have different values. This is the generic case.

Computing I_{ij}

Since the inertia tensor plays such an important role in rigid-body motion it is worth pausing to describe in more detail how it is in practice computed. Once given a rigid body's shape

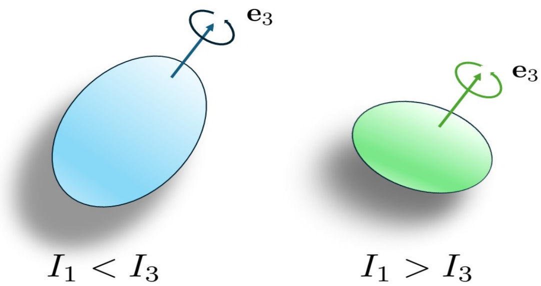

Figure 14. Examples of a symmetrical tops. These examples differ in the relative size of I_1 and I_3.

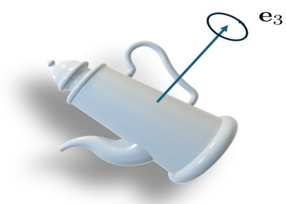

Figure 15. Example of an asymmetrical top.

there are two steps required to compute I_{ij}. First one finds the position of the centre of mass and then one performs the sum in (4.1.8) (or the integral in (4.1.15)).

This calculation is much simpler if the rigid body's mass is distributed symmetrically. For instance, if the body has a symmetry of reflection about a plane then the centre of mass and two of the principal axes must lie in this plane, with the other principal axis perpendicular to this plane.

A special case of an object with this kind of reflection symmetry is a planar object for which all of the atoms lie in a plane, in which case we may choose the principal axis perpendicular to be the 3 direction so that $\hat{x}_{3a} = 0$ for all atoms. Then expressions (4.1.20) imply

$$\mathcal{I}_1 = \sum_a m_a \hat{x}_{2a}^2 \,, \quad \mathcal{I}_2 = \sum_a m_a \hat{x}_{1a}^2 \quad \text{and} \quad \mathcal{I}_3 = \sum_a m_a (\hat{x}_{1a}^2 + \hat{x}_{2a}^2) \quad \text{(planar)} \tag{4.1.21}$$

and so for planar objects $\mathcal{I}_3 = \mathcal{I}_1 + \mathcal{I}_2$.

Another useful special case is the situation when an object is effectively one-dimensional: all atoms are lie along a single line. Such an object is called a *rotor*. Since this is a special case of a cylindrically symmetric situation two of the principal moments must be equal. Since it is also a special case of a planar object one of the principal moments should also be the sum of the other two. Choosing the atoms to lie along the 1 direction implies $\hat{x}_{2a} = \hat{x}_{3a} = 0$ and so (4.1.20) implies

$$\mathcal{I}_1 = 0 \,, \quad \mathcal{I}_2 = \mathcal{I}_3 = \sum_a m_a \hat{x}_{1a}^2 \qquad \text{(rotor)}\,, \tag{4.1.22}$$

which clearly does have two equal principal moments and satisfies $\mathcal{I}_3 = \mathcal{I}_1 + \mathcal{I}_2$. The vanishing of $\mathcal{I}_1$ expresses that there is no meaning to rotations about the line along which the atoms lie.

Worked example: Moment of inertia two small massive objects separated by a fixed distance

Consider the special case that all of the atoms of a rotor are crammed into two objects, A and B, of negligible size and masses

$$M_A = \sum_{a \in A} m_a \quad and \quad M_B = \sum_{b \in B} m_b \,, \tag{4.1.23}$$

separated by a distance L. Then the centre of mass is located on the line connecting bodies A and B at a distance R_A from A and a distance R_B from B, so $R_A + R_B = L$ and $M_A R_A = M_B R_B$. Solving for R_A and R_B implies $R_A = M_B L/M$ and $R_B = M_A L/M$ where $M = M_A + M_B$.

Choosing the line between the bodies to be the 1 direction we know from (4.1.22) that $\mathcal{I}_1 = 0$ and the other two principal moments share the common value

$$\mathcal{I}_2 = \mathcal{I}_3 = \sum_a m_a \hat{x}_{1a}^2 \simeq M_A R_A^2 + M_B R_B^2 = \frac{M_A M_B L^2}{M_A + M_B} = \mu L^2 \,, \tag{4.1.24}$$

where the approximate equality uses the negligible size of each body to write $\hat{x}_{1a} \simeq R_A$ for all $a \in A$ and $\hat{x}_{1a} \simeq R_B$ for all $a \in B$. The final equality uses the definition (1.2.9) of the reduced mass, μ, of two bodies.

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Worked example: Moment of inertia of a water molecule

Consider a water molecule, H_2O, consisting of two H atoms separated from an O atom by a distance ℓ. The angle between the two H-O bonds is $\theta \simeq 105^o$ (see Fig. 16). What is the moment of inertia of the molecule assuming the O atom has mass M and the H atoms have mass m and that the size of the atoms can be neglected?

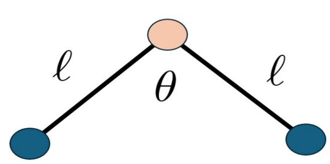

Figure 16. Sketch of a water molecule with two bonds of length ℓ separated by an angle $\theta \simeq 105^o$.

Because this is a special case of a planar system we can choose the plane defined by the atoms to be perpendicular to the third principal axis, in which case (4.1.21) implies $\mathcal{I}_3 = \mathcal{I}_1 + \mathcal{I}_2$. Furthermore, the reflection symmetry of the molecule within the 1-2 plane about the line midway between the two H atoms passing through the O atom (the vertical direction in Fig. 16) implies the other two principal axes lie perpendicular to this reflection plane (call this principal moment $\mathcal{I}_1$) and parallel to it (call this principal moment $\mathcal{I}_2$).

The principal moment $\mathcal{I}_1$ then is given by $\mathcal{I}_1 = 2mx^2$ where m is the H atom's mass and $x = \ell \sin \frac{\theta}{2}$ is the perpendicular distance to each H atom measured from the line of reflection symmetry. This gives

$$\mathcal{I}_1 = 2m\ell^2 \sin^2 \frac{\theta}{2} \,. \tag{4.1.25}$$

The principle moment $\mathcal{I}_2$ is given by $\mathcal{I}_2 = My_O^2 + 2my_H^2$ where M is the mass of the O atom and y denotes the vertical distance from the centre of mass to each of the two types of atoms. These satisfy $y_O + y_H = \ell \cos \frac{\theta}{2}$ and $My_O = 2my_H$, from which we infer $y_O = 2m\ell \cos \frac{\theta}{2}/(M+2m)$ and $y_H = M\ell \cos \frac{\theta}{2}/(M+2m)$, leading to

$$\mathcal{I}_2 = \frac{2mM\ell^2}{M+2m} \, \cos^2 \frac{\theta}{2} \,. \tag{4.1.26}$$

Planarity then implies

$$\mathcal{I}_3 = \mathcal{I}_1 + \mathcal{I}_2 = 2m\ell^2 \left(1 - \frac{2m}{M+2m} \, \cos^2 \frac{\theta}{2} \right) \,. \tag{4.1.27}$$

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Worked example: Moment of inertia of a uniform density rectangular parallelipiped with sides of length a, b and c.

Consider next a rigid body with the shape of a rectangular parallelipiped with sides of length a, b and c (see Fig. 17. The volume of such an object is $V = abc$. If its mass is M and its mass density, ρ, is constant then $\rho = M/V = M/(abc)$.

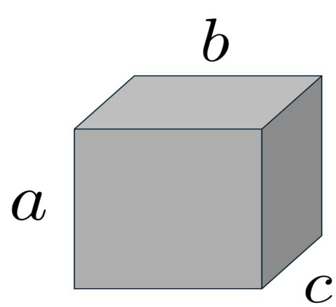

Figure 17. Sketch of a rigid parallelipiped whose sides have length a, b and c.

The symmetry of the problem implies the centre of mass sits at the object's geometrical centre, which is half-way along each side from any corner. The principal axes of inertia are also parallel to the sides of the parallelipiped, which we take to define the x, y and z axes (with the origin at the centre of mass). With these assumptions the expression for the moment of inertia tensor is found using (4.1.15), which in the present instance becomes

$$I_{ij} = \int_{-a/2}^{a/2} \mathrm{d}x \int_{-b/2}^{b/2} \mathrm{d}y \int_{-c/2}^{c/2} \mathrm{d}z \; \rho \Big[\delta_{ij} \left(x^2 + y^2 + z^2 \right) - x_i \, x_j \Big] \,. \tag{4.1.28}$$

As expected $I_{ij} = 0$ if $i \neq j$ because the result is proportional to the integral $\int_{-a/2}^{a/2} x \, \mathrm{d}x = 0$ (or its counterpart in the y or z direction). The diagonal elements are

$$\mathcal{I}_1 := I_{xx} = \frac{M}{abc} \int_{-a/2}^{a/2} \mathrm{d}x \; \rho \int_{-b/2}^{b/2} \mathrm{d}y \int_{-c/2}^{c/2} \mathrm{d}z \; (y^2 + z^2) = \frac{M}{12} (b^2 + c^2) \,, \tag{4.1.29}$$

and similarly

$$\mathcal{I}_2 := I_{yy} = \frac{M}{12} (a^2 + c^2) \qquad \text{and} \qquad \mathcal{I}_3 := I_{zz} = \frac{M}{12} (a^2 + b^2) \,. \tag{4.1.30}$$

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Exercise 4.1: Show that the principal moments of inertia of a uniform sphere of mass M and radius a are

$$\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}_3 = \tfrac{2}{5} \, M a^2 \,. \tag{4.1.31}$$

Exercise 4.2: Show that the principal moments of inertia of a circular cylinder of mass M, radius a and height h are

$$\mathcal{I}_1 = \mathcal{I}_2 = \tfrac{1}{4} M \Big(a^2 + \tfrac{1}{3} h^2 \Big) \quad \text{and} \quad \mathcal{I}_3 = \tfrac{1}{2} \, M a^2 \,, \tag{4.1.32}$$

where the axis of the cylinder is taken to lie along the 3 direction.

Exercise 4.3: Show that the centre of mass of a uniform cone of base radius a and height h lies a distance $\frac{3}{4} h$ from the cone's tip. Show that the principal moments of inertia of such a circular cone of mass M are

$$\mathcal{I}_1 = \mathcal{I}_2 = \tfrac{3}{20} M \Big(a^2 + \tfrac{1}{4} h^2 \Big) \quad \text{and} \quad \mathcal{I}_3 = \tfrac{3}{10} \, M a^2 \,, \tag{4.1.33}$$

where the axis of the cone is taken to lie along the 3 direction.

Exercise 4.4: Show that the principal moments of a uniform ellipsoid with mass M and semi-axes a, b and c are

$$\mathcal{I}_1 = \tfrac{1}{5} M \Big(b^2 + c^2 \Big) \quad \mathcal{I}_2 = \tfrac{1}{5} M \Big(a^2 + c^2 \Big) \quad \text{and} \quad \mathcal{I}_3 = \tfrac{1}{5} \, M \Big(a^2 + b^2 \Big) \,, \tag{4.1.34}$$

where the axes of the ellipsoid are taken to lie along the 1, 2 and 3 directions.

4.1.3 Angular Momentum

For later purposes it is also useful to express the angular momentum in terms of the angular velocity $\mathbf{\Omega}$, where the angular momentum $\mathbf{J}$ is computed relative to the origin of coordinates (which is not assumed to be the rigid body's centre of mass). Writing $\mathbf{r}_a = \mathbf{R} + \mathbf{s}_a$ in the definition (1.4.8) gives:

$$\mathbf{J} = \sum_a m_a \, \mathbf{r}_a \times \mathbf{v}_a = \mathbf{R} \times \sum_a m_a \, \mathbf{v}_a + \mathbf{M} \tag{4.1.35}$$

where the first term can be written

$$\mathbf{L} := \mathbf{R} \times \mathbf{P} = M\mathbf{R} \times \dot{\mathbf{R}} \tag{4.1.36}$$

where $\mathbf{P} = \sum_a m_a \mathbf{v}_a = M\mathbf{V}$ is the centre of mass momentum, and so represents the 'orbital' angular momentum due to the overall motion of the body's centre of mass.

The second term, $\mathbf{M}$, of (4.1.35) is the 'intrinsic' angular momentum associated with the rigid body's rotation about its own centre of mass, and is given by

$$\begin{aligned}\mathbf{M} := \sum_a m_a \, \mathbf{s}_a \times \mathbf{v}_a &= \sum_a m_a \, \mathbf{s}_a \times \Big(\mathbf{V} + \mathbf{\Omega} \times \mathbf{s}_a\Big) = \sum_a m_a \, \mathbf{s}_a \times \Big(\mathbf{\Omega} \times \mathbf{s}_a\Big) \\ &= \sum_a m_a \Big[\mathbf{s}_a^2 \, \mathbf{\Omega} - (\mathbf{s}_a \cdot \mathbf{\Omega})\mathbf{s}_a\Big]\end{aligned} \tag{4.1.37}$$

where the second equality of the first line uses (4.1.4) to eliminate $\mathbf{v}_a$ in terms of the centre-of-mass velocity $\mathbf{V}$ and the velocity $\mathbf{\Omega} \times \mathbf{s}_a$ due to rotation around the centre of mass and the third equality of the first line uses the identity $\sum_a m_a \mathbf{s}_a = 0$ (see eq. (1.4.4)) to drop the term involving $\mathbf{V}$. The second line follows after using the vector identity (A.3.14) and shows how $\mathbf{M}$ is in general not parallel to $\mathbf{\Omega}$.

Eq. (4.1.37) also shows the components of $\mathbf{M}$ are linearly related to the components of $\mathbf{\Omega}$, and the matrix that appears in this linear relation is one that we have seen before: the moment of inertia tensor defined in (4.1.8). That is, in components (4.1.37) states

$$M_i = I_{ij} \Omega_j \quad \text{which in matrix form becomes} \quad \mathbf{M} = \mathbb{I} \, \mathbf{\Omega} \, . \tag{4.1.38}$$

These expressions allow eq. (4.1.7) for the rotational kinetic energy to be rewritten

$$E_{\text{rot}} = \tfrac{1}{2} \, \mathbf{\Omega} \cdot \mathbf{M} \, . \tag{4.1.39}$$

Although $\mathbf{M}$ is in general not parallel to $\mathbf{\Omega}$ there are a few specific situations where they can be parallel. For a generically shaped object this happens if the rotation axis is parallel to one of the principal axes of inertia (*i.e.* it is parallel to an eigenvector of the matrix $\mathbb{I}$). In this case $\mathbb{I} \, \mathbf{\Omega} = \mathcal{I} \, \mathbf{\Omega}$ with eigenvalue $\mathcal{I}$ and so the angular momentum becomes $\mathbf{M} = \mathcal{I} \, \mathbf{\Omega}$, which is parallel to $\mathbf{\Omega}$ as claimed.

For a spherical top (for which $I_{ij} = \mathcal{I} \, \delta_{ij}$ is proportional to the unit matrix) it is true that every direction is a principal axis and so $\mathbf{M}$ is always parallel to $\mathbf{\Omega}$, regardless of the direction of $\mathbf{\Omega}$.

4.2 Rotation *not* about the centre of mass

To this point we have seen that an infinitesimal rotation defines a direction $\delta\boldsymbol{\phi}$, in terms of which we define the instantaneous angular velocity $\boldsymbol{\Omega} = \lim_{\delta t\to 0} \delta\boldsymbol{\phi}/\delta t$. But a in infinitesimal rotation contains more than just a direction: it also defines a line of points – the rotation's *axis* – about which an object pivots and so whose atom's do not move at all during the rotation.

We have also seen that the general motion of a rigid body can be regarded as instantaneously being a combination of the translation of the centre of mass plus a rotation *about the centre of mass*, and because of this we have considered only rotation axes that at any given instant pass through the object's centre of mass. This section extends the discussion of the kinematics of rotation to the often useful situation when the axis of rotation is not restricted in this way.

To this end, suppose we go back and repeat the arguments leading to eq. (4.1.4) (repeated here for ease of reference)

$$\mathbf{v} = \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{s}\,, \tag{4.2.1}$$

but this time compare $\mathbf{v}$ for the same point to the velocity $\mathbf{V}'$ and angular velocity $\boldsymbol{\Omega}'$ for rotations relative to a reference point O' that also moves with the body but is displaced from the centre of mass by an amount $\mathbf{a}$.

In this case $\mathbf{r} = \mathbf{R}' + \mathbf{s}'$, where $\mathbf{R}' = \mathbf{R} + \mathbf{a}$ is the position of O' and $\mathbf{s}'$ is the position $\mathbf{r}$ as seen by an observer using O' as the origin of coordinates. The same argument made in §4.1.1 then implies the velocities and angular velocity measured in this new frame are related by

$$\mathbf{v} = \mathbf{V}' + \boldsymbol{\Omega}' \times \mathbf{s}'\,. \tag{4.2.2}$$

where $\mathbf{V}' = \mathrm{d}\mathbf{R}'/\mathrm{d}t$ is the velocity of O' and $\boldsymbol{\Omega}' = \mathrm{d}\boldsymbol{\phi}'/\mathrm{d}t$ is the angular velocity for the rotation $\delta\mathbf{s}' = \delta\boldsymbol{\phi}' \times \mathbf{s}'$ about O'. We do not assume that $\boldsymbol{\Omega}'$ is the same as the angular velocity $\boldsymbol{\Omega}$ for the observer at the centre of mass. We wish to compute what $\boldsymbol{\Omega}'$ is relative to $\boldsymbol{\Omega}$.

To do so notice that on one hand O' is just another point on the rigid body and as a result its velocity $\mathbf{V}'$ must also satisfy (4.2.1):

$$\mathbf{V}' = \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{a}\,, \tag{4.2.3}$$

where $\mathbf{a}$ is the displacement of O' relative to the body's centre of mass. But since O' is displaced from the centre of mass by a fixed vector $\mathbf{a}$ then $\mathbf{s} = \mathbf{s}' + \mathbf{a}$, which when used in (4.2.1) implies

$$\mathbf{v} = \mathbf{V} + \boldsymbol{\Omega} \times (\mathbf{s}' + \mathbf{a})\,. \tag{4.2.4}$$

Comparing this with (4.2.2) and using (4.2.3) gives the answer we seek:

$$\boldsymbol{\Omega}' = \boldsymbol{\Omega}\,. \tag{4.2.5}$$

We see in this way that the instantaneous angular rotation of a system of coordinates fixed to move with the rigid body is the same *regardless of the origin about which the rotation is measured.* In that sense the angular velocity $\mathbf{\Omega}$ can be regarded to be a property of the body, much like the centre of mass velocity $\mathbf{V}$ (but *unlike* the instantaneous velocity $\mathbf{v}$ of any particular point of the body).

It can sometimes happen that the motion $\mathbf{v}$ can be regarded as a pure rotation relative to some position O'. In particular if it happens to be true at some instant that $\mathbf{V} \cdot \mathbf{\Omega} = 0$ (*i.e.* they are perpendicular) then (4.1.4) shows that $\mathbf{v} \cdot \mathbf{\Omega} = 0$ is also true for the instantaneous velocity of any point on the body. The converse is also true: if there is a point for which $\mathbf{v} \cdot \mathbf{\Omega} = 0$ then (4.1.4) shows it is also true for $\mathbf{V}$ and so must be true for the velocity for all points on the rigid body. In this case because $\mathbf{V} \cdot \mathbf{\Omega} = 0$ there must always exist a $\mathbf{b}$ such that $\mathbf{V} = -\mathbf{\Omega} \times \mathbf{b}$. Consequently the reference point O' displaced from the centre of mass by $\mathbf{b}$ (that possibly might not lie inside the body) has a velocity that vanishes because (4.2.3) implies $\mathbf{V}' = \mathbf{V} + \mathbf{\Omega} \times \mathbf{b} = 0$. Eq. (4.2.2) together with (4.2.5) then imply the velocity relative to this frame is $\mathbf{v} = \mathbf{\Omega} \times \mathbf{s}'$: that is to say there is a frame within which the motion is instantaneously a pure rotation.

A similar argument in the case where $\mathbf{V}$ and $\mathbf{\Omega}$ are not orthogonal shows that a frame can be chosen for which $\mathbf{V}$ and $\mathbf{\Omega}$ are parallel and so the motion is instantaneously a combination of a rotation about some axis together with a translation along that same axis.

4.2.1 Moments of Inertia for Displaced Axes

The quantities $\mathbf{V}'$ and $\mathbf{\Omega}$ provide an alternative six velocity components needed to describe rigid-body motion, though they are often less convenient because in general $\sum_a m_a \mathbf{s}'_a \neq 0$ (unlike for displacements $\mathbf{s}_a$ relative to the centre of mass) and so expressions like (4.1.5) and (4.1.6) for the kinetic energy need not cleanly divide up into contributions from $\mathbf{V}'$ and $\mathbf{\Omega}$ separately.

But some problems involve objects rotating about axes that do not pass through their centre of mass, and for which the various forces in the problem hold the axis fixed (so $\mathbf{V}' = 0$). In this case it does not matter that the kinetic energy is not just a sum of a $\mathbf{V}'$-dependent piece and and a $\mathbf{\Omega}$-dependent piece, and it can be useful to work in a more general reference frame. In such a case the kinetic energy is again quadratic in the components Ω_i:

$$E_{\rm rot} = \tfrac12 \sum_a m_a (\dot{\mathbf{s}}')^2 = \tfrac12 \sum_a m_a \Big(\mathbf{\Omega} \times \mathbf{s}'\Big)^2 = \tfrac12 I'_{ij} \Omega_i \Omega_j \,, \tag{4.2.6}$$

where

$$I'_{ij} := \sum_a m_a \Big[\delta_{ij} \, (\mathbf{s}'_a)^2 - x'_{ai} \, x'_{aj}\Big] \,. \tag{4.2.7}$$

The momentum of inertia tensor I'_{ij} about a point displaced relative to the centre of mass by $\mathbf{a}$ can be related to the moment of inertia tensor I_{ij} for rotations through the centre of mass since direct use of the definitions together with $\mathbf{s}'_a = \mathbf{s}_a + \mathbf{a}$ and the property $\sum_a m_a \mathbf{s}_a = 0$ shows that

$$I'_{ij} = I_{ij} + M\Big[\delta_{ij}\,\mathbf{a}^2 - a_i\,a_j\Big]\,, \tag{4.2.8}$$

where (as usual) $M = \sum_a m_a$. This expression can be useful, particularly if I'_{ij} is easier to compute than I_{ij}.

Worked example: Moment of inertia of a uniform density rectangular parallelipiped computed relative to one of the corners.

For the rectangular rigid body with sides of length a, b and c shown in Fig. 17 we can compute the moment of inertia tensor I'_{ij} relative to one of the corners rather than the centre of mass.

In this case the problem below eq. (4.1.27) gives the moment of inertia relative to the centre of mass in rectangular coordinates adapted to the symmetry axes of the body:

$$\mathbb{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} = \frac{M}{12}\begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}\,. \tag{4.2.9}$$

The displacement vector from the centre of mass to one of the corners is

$$\mathbf{a} = \tfrac{1}{2}\Big(a\,\mathbf{e}_x + b\,\mathbf{e}_y + c\,\mathbf{e}_z\Big)\,, \tag{4.2.10}$$

and so

$$\Big\{\delta_{ij}\mathbf{a}^2 - a_i a_j\Big\} = \frac{1}{4}\begin{pmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{pmatrix}\,. \tag{4.2.11}$$

Therefore eq. (4.2.8) evaluates to

$$\mathbb{I}' = \begin{pmatrix} \tilde{I}_{xx} & I'_{xy} & I'_{xz} \\ I'_{yx} & I'_{yy} & I'_{yz} \\ I'_{zx} & I'_{zy} & I'_{zz} \end{pmatrix} = M\begin{pmatrix} \frac13(b^2+c^2) & -\frac14 ab & -\frac14 ac \\ -\frac14 ab & \frac13(a^2+c^2) & -\frac14 bc \\ -\frac14 ac & -\frac14 bc & \frac13(a^2+b^2) \end{pmatrix}\,. \tag{4.2.12}$$

These expressions become the standard result from elementary mechanics for a uniform rod of mass M and length L in the limit that $b \to 0$ and $c \to 0$ with $a \to L$ and M fixed. For instance the moment moment of inertia is $\mathfrak{I} = I_{yy} = I_{xx} = \frac{1}{12}ML^2$ for rotations about an axis perpendicular to the rod, passing through its centre of mass, but is $\mathfrak{I}' = I'_{xx} = I'_{yy} = \frac13 ML^2$ for rotations about an axis perpendicular to the rod's endpoint.

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Exercise 4.5: Show by direct integration that the principal moments of inertia of a circular cone of mass M, base radius a and height h when taken about the point at the tip of the cone are

$$\mathcal{I}'_1 = \mathcal{I}'_2 = \tfrac35 M\Big(h^2 + \tfrac14 a^2\Big) \quad \text{and} \quad \mathcal{I}'_3 = \tfrac{3}{10}\,Ma^2\,, \tag{4.2.13}$$

where the axis of the cone is taken to lie along the 3 direction. Verify that this satisfies (4.2.8) when compared with the result (4.1.33) evaluated about the centre of mass.

4.3 Dynamics of Isolated Rigid Bodies

The next step is to describe how the position and rotation of a rigid body responds to applied forces. At one level things are particularly simple for rigid bodies: only six degrees of freedom are needed to completely describe their motion and so only six equations of motion are required to predict how they move. These are conveniently given by the equations expressing conservation of linear and angular momentum.

That is, the general expression for the motion of the centre of mass goes back to eq. (1.3.3), which states

$$\dot{\mathbf{P}} = \mathbf{F}^{\rm ext}_{\rm tot} \,, \tag{4.3.1}$$

where $\mathbf{P} = M\mathbf{V}$ is the rigid body's total linear momentum and $\mathbf{F}^{\rm ext}_{\rm tot}$ is the sum of all of the *external* forces acting on it. Conveniently – as argued at length in §1.3.1 – internal interatomic forces cancel out in this expression and so play no role in the evolution of $\mathbf{R}$.

The corresponding evolution equation for the angular degrees of freedom are those given in (1.4.9), which state

$$\dot{\mathbf{J}} = \boldsymbol{\tau}^{\rm int}_{\rm tot} + \boldsymbol{\tau}^{\rm ext}_{\rm tot} \,, \tag{4.3.2}$$

where

$$\boldsymbol{\tau}^{\rm int}_{\rm tot} := \sum_{a>b} (\mathbf{r}_a - \mathbf{r}_b) \times \mathbf{F}_{ab} \quad \text{and} \quad \boldsymbol{\tau}^{\rm ext}_{\rm tot} := \sum_a \mathbf{r}_a \times \mathbf{F}^{\rm ext}_a \tag{4.3.3}$$

respectively are the net torques due to the internal and externally applied forces. If interatomic forces all come from rotation-invariant laws of physics[11] (as all current evidence suggests they should) we know from Noether's theorem – *c.f.* §2.3.3 – that angular momentum must be conserved for isolated bodies and so the net internal torque $\boldsymbol{\tau}^{\rm int}_{\rm tot}$ must vanish, leaving

$$\dot{\mathbf{J}} = \boldsymbol{\tau}^{\rm ext}_{\rm tot} \,. \tag{4.3.4}$$

Both (4.3.1) and (4.3.4) are necessarily written in an inertial frame (which in particular cannot be the rotating body frame that moves with the rigid body). All torques and angular momenta are computed relative to the body's centre of mass.

In this section we restrict to the case of isolated rigid bodies and so temporarily assume $\mathbf{F}^{\rm ext}_{\rm tot} = \boldsymbol{\tau}^{\rm ext}_{\rm tot} = 0$, returning to the more general case in later sections.

4.3.1 View from an Inertial Frame

In the absence of external forces the centre of mass does not accelerate and so moves with a constant velocity. Since the focus here is on the rotational motion there is no loss of generality in choosing an inertial frame for which the centre of mass does not move, in which case the body's only angular momentum is the intrinsic angular momentum: $\mathbf{J} = \mathbf{M}$ (compare with

[11] This includes, but is not restricted to, central forces like Coulomb's law, for which each term in the sum in $\boldsymbol{\tau}^{\rm int}_{\rm tot}$ separately vanishes.

the general case, (4.1.35)). In situations where the rigid body moves in complicated ways we imagine choosing the inertial frame to be instantaneously at rest relative to the rigid body, with axes instantaneously aligned with body frame bases (such as the principal axes).

With these assumptions (4.3.4) implies

$$\dot{\mathbf{M}} = 0\,, \tag{4.3.5}$$

whose implications we seek for the body's angular motion around its centre of mass. Clearly $\mathbf{M}$ is constant in an inertial frame. It is tempting to argue that because $\mathbf{M} = \mathbb{I}\,\boldsymbol{\Omega}$ the instantaneous angular velocity $\boldsymbol{\Omega}$ must also be constant, but this is in general *not* true.

The fallacy in this argument lies in the assumption that I_{ij} cannot change in time in the inertial frame. Although the rigidity of a rigid body does ensure that the components $\hat{I}_{ij}$ of the inertia tensor are time-independent when expressed in terms of body axes that move with the object, it in general does not ensure I_{ij} remains constant in the inertial frame because the boundaries of the object generically move in this frame as the object rotates. (For instance for a rod rotating about its centre of mass in the x-y plane the components I_{xx} and I_{yy} are different when the rod is parallel to the x axis from what they are when the rod is parallel to the y axis.)

Alternatively, the time dependence in an inertial frame is given by a time-dependent rotation: $I_{ij}(t) = R_{im}(t)\,\hat{I}_{mn}\,R_{jn}(t)$ – or equivalently $\mathbb{I}(t) = R(t)\,\hat{\mathbb{I}}\,R^T(t)$ for some rotation matrix $R(t)$, where $\hat{I}_{ij}$ are the time-independent elements of the inertia tensor computed in a frame that moves with the rigid body. But this means that the eigenvalues, $\mathcal{I}_i$, of $I_{ij}(t)$ must remain time-independent even as $I_{ij}(t)$ varies with time in an inertial frame. To see why notice that under a rotation $\mathbb{I}(t) = R(t)\,\hat{\mathbb{I}}\,R^T(t)$ and a vector transforms as $\mathbf{u}(t) = R(t)\,\hat{\mathbf{u}}$, and so $\mathbb{I}(t)\,\mathbf{u}(t) = R(t)\,\hat{\mathbb{I}}\,\hat{\mathbf{u}}$ and so the eigenvalue condition

$$\hat{I}_{ij}\,\hat{u}_j = \mathcal{I}\,\hat{u}_i \quad \text{becomes} \quad I_{ij}\,u_j = \mathcal{I}\,u_i\,, \tag{4.3.6}$$

after a rotation with the eigenvalue $\mathcal{I}$ unchanged.

So the motion implied by $\dot{\mathbf{M}} = 0$ need not be as boring as one might have thought because it can allow $\boldsymbol{\Omega}(t)$ to depend on time. Conservation of kinetic energy $K = \frac{1}{2}\boldsymbol{\Omega}\cdot\mathbf{M}$ then shows that if $\boldsymbol{\Omega}$ does evolve it does so with a fixed dot product with $\mathbf{M}$.

Spherical Top

In the special case of a spherical top all three principal moments of inertia are equal: $\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}_3 = \mathcal{I}$. Because the eigenvalues of I_{ij} are frame-independent they must also be time-independent because I_{ij} is time-independent in a 'body' frame that moves with the rigid body. It follows that in this case $\mathbf{M} = \mathcal{I}\,\boldsymbol{\Omega}$ in any frame and so $\mathbf{M}$ and $\boldsymbol{\Omega}$ are parallel at all times, where $\boldsymbol{\Omega}$ defines the axis of rotation passing through the body's centre of mass.

In this case $\boldsymbol{\Omega}$ must also be a constant vector and so the rotation axis points in a fixed direction with a constant angular speed. In the absence of external forces a spinning sphere just sits there spinning at constant angular speed with its rotation axis pointed in a fixed spatial direction. The stability of this type of rotaional motion is the underlying fact behind gyroscope design.

Rotation about a Principal Axis

There is a special case where more asymmetric objects move just as simply as do spherical tops: when the rotation axis is chosen to be one of the body's principal axes.

Suppose we work in an inertial frame and $\mathbf{\Omega}(t_0)$ is chosen initially parallel to one of the eigenvectors of $I_{ij}(t_0)$ at that time. If $\mathcal{I}_\star$ is the corresponding eigenvalue – which, as argued above, must be time-independent – then the angular momentum at this initial time is

$$M_i(t_0) = I_{ij}(t_0)\Omega_j(t_0) = \mathcal{I}_\star \Omega_i(t_0)\,, \tag{4.3.7}$$

so $\mathbf{M}$ is also parallel to the same principal direction (and so is parallel to the same eigenvector of I_{ij}.

But time evolution of the system is found by rotating it using the matrix $R(t)$ whose instantaneous angular velocity is $\mathbf{\Omega}$ (we see in more detail how $R(t)$ and $\mathbf{\Omega}$ are related to one another in §4.4). This rotation does not alter the eigenvalue condition $\mathcal{I}_{ij}M_j = \mathcal{I}_\star M_i$, which is basis independent, so the solution to $\dot{\mathbf{M}} = 0$ is to have $\mathbf{\Omega}(t)$ to remain parallel to the same principal axis for all time, and so (4.3.7) remains true for all t.

The argument does not depend on which principal axis was used so if any rigid body is initially spun about any principal axis then $\mathbf{\Omega}$ is constant and the body rotates at constant angular speed about this axis, with a fixed direction in space.

Symmetric Top

The story is more interesting when less symmetrical objects are initially spun about axes that are not aligned with a principal axis. To explore this in more detail consider first a symmetric top, for which only two principal moments of inertia are equal. In what follows we denote the body's principal axes by $\hat{\mathbf{e}}_i(t)$ and choose the labels $i = 1, 2, 3$ in such a way that $\mathcal{I}_1 = \mathcal{I}_2 \neq \mathcal{I}_3$, so $\hat{\mathbf{e}}_3$ points along the symmetry axis in cases where the object is cylindrically symmetric. Because $\mathbf{\Omega}$ is not initially chosen to lie along one of the principal axes it must have a component both parallel to the symmetry axis and a component perpendicular to it.

Some useful conclusions follow directly from conservation of energy and angular momentum. These imply that $\mathbf{M}$ and $\mathbf{M}\cdot\mathbf{\Omega}$ are both time independent in an inertial frame. Recall also that rotational scalars like $|\mathbf{M}|$ and $\mathbf{M}\cdot\mathbf{\Omega}$ are frame independent, so both $|\mathbf{M}|$ and $\mathbf{M}\cdot\mathbf{\Omega}$ must also be time-independent in the body frame. The principal moments $\mathcal{I}_1$ and $\mathcal{I}_3$ are also time-independent, and these facts allow some inferences to be drawn about the components of $\mathbf{\Omega}$ in the body frame.

To see why, start with the general relation (4.1.38) evaluated in the body frame:

$$\mathbf{\Omega} = \Omega_1\hat{\mathbf{e}}_1 + \Omega_2\hat{\mathbf{e}}_2 + \Omega_3\hat{\mathbf{e}}_3 \quad \text{and so} \quad \mathbf{M} = \mathcal{I}_1\Big(\Omega_1\,\hat{\mathbf{e}}_1 + \Omega_2\,\hat{\mathbf{e}}_2\Big) + \mathcal{I}_3\,\Omega_3\,\hat{\mathbf{e}}_3\,, \tag{4.3.8}$$

where the expression for $\mathbf{M}$ uses the symmetric-top result $\mathcal{I}_1 = \mathcal{I}_2$. The constant values for $|\mathbf{M}|^2 = \mathcal{I}_1^2\Big(\Omega_1^2 + \Omega_2^2\Big) + \mathcal{I}_3^2\Omega_3^2$ and $\mathbf{M}\cdot\boldsymbol{\Omega} = \mathcal{I}_1\Big(\Omega_1^2+\Omega_2^2\Big) + \mathcal{I}_3\Omega_3^2$ therefore imply

$$\Omega_3 \quad \text{and} \quad \Omega_\perp^2 := \Omega_1^2 + \Omega_2^2 \quad \text{are both time-independent.} \tag{4.3.9}$$

From this it follows that $M_3 = \mathcal{I}_3\Omega_3$ and $|\boldsymbol{\Omega}|$ are also both time-independent. Finally, time-independence of $|\boldsymbol{\Omega}|$ and $\mathbf{M}\cdot\boldsymbol{\Omega} = |\mathcal{M}||\boldsymbol{\Omega}|\cos\varphi$ together imply that the angle φ between $\mathbf{M}$ and Ω is time-independent. Similarly the angle ϑ between $\mathbf{M}$ and $\hat{\mathbf{e}}_3$ is also independent since $\mathbf{M}\cdot\hat{\mathbf{e}}_3 = M_3 = |\mathbf{M}|\cos\vartheta$.

Ω_1 and Ω_2 are not themselves separately time-independent, however, and this can most easily be seen by making a convenient choice for the body frame axes (which are not uniquely defined because the condition $\mathcal{I}_1 = \mathcal{I}_2$ means any two perpendicular directions in the 1-2 plane are principal axes). Although it is tempting to try to choose the principal axes $\hat{\mathbf{e}}_1(t)$ and $\hat{\mathbf{e}}_2(t)$ so that $\hat{\mathbf{e}}_2(t)$ is aways perpendicular to $\boldsymbol{\Omega}$, this cannot be done if $\boldsymbol{\Omega}$ moves in the body frame (as it in general does, as it turns out). But we *can* arrange $\boldsymbol{\Omega}(t)$ to be perpendicular to $\hat{\mathbf{e}}_2$ at any particular instant in time, if we keep in mind that $\boldsymbol{\Omega}$ and $\hat{\mathbf{e}}_2$ in general will not stay perpendicular as time evolves.

At an instant where $\boldsymbol{\Omega}$ is perpendicular to $\hat{\mathbf{e}}_2$ eqs. (4.3.8) simplify to

$$\boldsymbol{\Omega} = \Omega_1\hat{\mathbf{e}}_1 + \Omega_3\hat{\mathbf{e}}_3 \quad \text{and so} \quad \mathbf{M} = \mathcal{I}_1\,\Omega_1\,\hat{\mathbf{e}}_1 + \mathcal{I}_3\,\Omega_3\,\hat{\mathbf{e}}_3\,, \tag{4.3.10}$$

which shows that all three of $\mathbf{M}$, $\boldsymbol{\Omega}$ and $\hat{\mathbf{e}}_3$ are coplanar at this instant. Because the instant chosen was arbitrary these three vectors must be coplanar at all times. To understand how the symmetric top moves in the inertial frame it is useful to focus our attention on those atoms that lie along the symmetry axis of the body, for which $\mathbf{s}_{\rm axis} \propto \hat{\mathbf{e}}_3$. The velocity for any atom that sits on the symmetry axis is given in the rest frame of the centre of mass by (4.2.1), which in this instance becomes

$$\mathbf{v}_{\rm axis}(t) = \dot{\mathbf{s}}_{\rm axis}(t) = \boldsymbol{\Omega}(t)\times\mathbf{s}_{\rm axis}(t)\,. \tag{4.3.11}$$

Expression (4.3.11) would have vanished if $\boldsymbol{\Omega}$ had been parallel to the symmetry axis, but it instead now is the equation for circular motion – see *e.g.* eq. (3.2.5) – and so shows that each point of the symmetry axis has a velocity that is instantaneously perpendicular to the plane containing $\boldsymbol{\Omega}$, $\mathbf{M}$ and $\mathbf{s}_{\rm axis} \propto \hat{\mathbf{e}}_3$. This means any point on the symmetry axis traces out a circle in the plane perpendicular to $\mathbf{M}$, as illustrated in Fig. 18: in the inertial frame the vector $\hat{\mathbf{e}}_3(t)$ *precesses* about the direction of $\mathbf{M}$.

The upshot is the direction $\hat{\mathbf{e}}_3(t)$ in the inertial frame does *not* point in a fixed direction (when $\mathbf{M}$ and $\boldsymbol{\Omega}$ are not parallel). It instead changes direction with time as the axis of symmetry of the symmetric top precesses about the direction of $\mathbf{M}$. Because $\mathbf{M}$, $\boldsymbol{\Omega}$ and $\hat{\mathbf{e}}_3$ must all be coplanar this means that $\boldsymbol{\Omega}$ must also precess around $\mathbf{M}$ with the same angular speed as $\hat{\mathbf{e}}_3$. That is,

$$\boldsymbol{\Omega}(t) = a\,\mathbf{M} + b\,\hat{\mathbf{e}}_3(t) \tag{4.3.12}$$

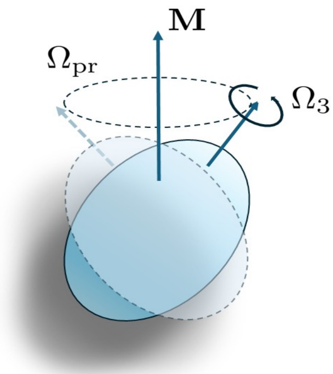

Figure 18. Procession of an isolated symmetrical top.

for some a and b. Comparing this with (4.3.8) implies

$$a = \frac{1}{\mathcal{I}_1} \quad \text{and} \quad b = \frac{(\mathcal{I}_1 - \mathcal{I}_3)M_3}{\mathcal{I}_1\mathcal{I}_3} = \left(\frac{\mathcal{I}_1 - \mathcal{I}_3}{\mathcal{I}_1}\right)\Omega_3 \,. \tag{4.3.13}$$

In particular, both a and b are constants.

The precession velocity is found by using (4.3.12) in (4.3.11), and using the kinematics of circular motion. For a point on the axis a distance ℓ from the centre of mass we have $\mathbf{s}_{\rm axis}(t) = \ell\,\hat{\mathbf{e}}_3(t)$ and so (4.3.12) with (4.3.11) together give

$$v_{\rm axis} := |\dot{\mathbf{s}}_{\rm axis}(t)| = \ell\,\Omega_\perp = \frac{\ell}{\mathcal{I}_1}|\mathbf{M}\times\hat{\mathbf{e}}_3(t)| = \frac{\ell\mathcal{M}\sin\vartheta}{\mathcal{I}_1}\,, \tag{4.3.14}$$

which uses the definition (4.3.9) for $\Omega_\perp$ and denotes $\mathcal{M} := |\mathbf{M}|$ while ϑ is the time-independent angle between $\mathbf{M}$ and $\hat{\mathbf{e}}_3$. We see that a point on the symmetry axis a distance ℓ from the centre of mass moves in a circle of radius $r_{\rm axis} = \ell\sin\vartheta$ with constant speed $v_{\rm axis} = \ell\mathcal{M}\sin\vartheta/\mathcal{I}_1$ and so its angular speed is

$$\Omega_{\rm pr} = \frac{v_{\rm axis}}{r_{\rm axis}} = \frac{\mathcal{M}}{\mathcal{I}_1}\,. \tag{4.3.15}$$

4.3.2 Euler's Equations

Part of what made the above discussion of precession cumbersome was an underlying tension between the simplicity of Newton's laws in an inertial frame and the simplicity of the properties of the rigid body – like its principal axes and time-dependence of I_{ij} – when expressed in a frame that moves with the object's motion.

This suggests re-examining the problem to see if it is easier using a body-centred frame. The key step for doing so is eq. (3.2.4) (repeated for convenience here)

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \frac{\partial\mathbf{X}}{\partial t} + \boldsymbol{\Omega}\times\mathbf{X}\,. \tag{4.3.16}$$

that expresses the rate of change $\mathrm{d}\mathbf{X}/\mathrm{d}t$ of an *arbitrary* vector $\mathbf{X}$ as seen in an inertial frame in terms of the rate of change $\partial\mathbf{X}/\partial t$ seen in a reference frame that rotates with an instantaneous angular velocity $\boldsymbol{\Omega}$. When applied to the angular momentum $\mathbf{M}$ – for which $\mathrm{d}\mathbf{M}/\mathrm{d}t = 0$ in an inertial frame – this implies the evolution of $\mathbf{M}$ for an isolated rigid body becomes

$$\frac{\partial \mathbf{M}}{\partial t} + \boldsymbol{\Omega} \times \mathbf{M} = 0\,. \tag{4.3.17}$$

once expressed in a rotating body frame.

A great advantage of a body frame is the components I_{ij} are time-independent (for a rigid body) and so $\mathbf{M}$ can be eliminated to give an evolution equation involving only $\boldsymbol{\Omega}$. This is simplest if evaluated using a body reference frame that is adapted to the rigid body's principal moments, in which case

$$\mathcal{I}_1 \frac{\partial \Omega_1}{\partial t} + (\mathcal{I}_3 - \mathcal{I}_2)\Omega_2\Omega_3 = 0 \tag{4.3.18a}$$

$$\mathcal{I}_2 \frac{\partial \Omega_2}{\partial t} + (\mathcal{I}_1 - \mathcal{I}_3)\Omega_1\Omega_3 = 0 \tag{4.3.18b}$$

$$\mathcal{I}_3 \frac{\partial \Omega_3}{\partial t} + (\mathcal{I}_2 - \mathcal{I}_1)\Omega_1\Omega_2 = 0\,, \tag{4.3.18c}$$

a set of equations called *Euler's equations* for an isolated rigid body. We next use these to reproduce the discussion given above for the evolution of rigid bodies with differing amounts of symmetry.

Spherical Top

The most symmetric example is the spherical top, for which $\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}_3 = \mathcal{I}$ and so eqs. (4.3.18) reduce to the statements that the components of angular velocity in the principal basis are constants:

$$\frac{\partial \Omega_1}{\partial t} = \frac{\partial \Omega_2}{\partial t} = \frac{\partial \Omega_3}{\partial t} = 0 \qquad \text{(spherical top)}. \tag{4.3.19}$$

The angular velocity vector $\boldsymbol{\Omega}$ therefore points in a constant direction relative to the axes that rotate with the rigid body. Using $\partial\boldsymbol{\Omega}/\partial t = 0$ in (4.3.16) applied to $\mathbf{X} = \boldsymbol{\Omega}$ then implies $\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t = 0$, in agreement with the earlier result.

Rotation about a Principal Axis

Another simple situation is the case where the rigid body need not be symmetric but the rotation is initially chosen to be along one of the principal axes. (We choose it here to be $\hat{\mathbf{e}}_1$ for the sake of argument, but the argument is the same for either of the others.)

If $\Omega_2 = \Omega_3 = 0$ and $\Omega_1 = \omega$ at an initial time $t = t_0$ then eqs. (4.3.18) show that $\partial\Omega_i/\partial t = 0$ for all three angular velocity components at this time as well. This ensures that the initial values do not change and so $\Omega_2 = \Omega_3 = 0$ for all time and $\Omega_1 = \omega$ for all times. This reproduces our earlier findings that an arbitrary rigid body initially rotated about one of the principal axes produces a constant angular velocity that remains along the same principal axis for all times.

Symmetric Top

For a symmetric top only two of the principal moments are equal, which we choose to be $\mathcal{I}_1 = \mathcal{I}_2 \neq \mathcal{I}_3$, so $\hat{\mathbf{e}}_3$ is the top's symmetry axis. In this case eqs. (4.3.18) reduce to:

$$\mathcal{I}_1 \frac{\partial \Omega_1}{\partial t} + (\mathcal{I}_3 - \mathcal{I}_1)\Omega_2\Omega_3 = 0 \,, \quad \mathcal{I}_1 \frac{\partial \Omega_2}{\partial t} + (\mathcal{I}_1 - \mathcal{I}_3)\Omega_1\Omega_3 = 0 \,, \tag{4.3.20}$$

and

$$\mathcal{I}_3 \frac{\partial \Omega_3}{\partial t} = 0 \,. \tag{4.3.21}$$

Equation (4.3.21) tells us the rotation rate about the symmetry direction, Ω_3, is constant. Using this in eqs. (4.3.20) then gives a pair of coupled linear equations that can be decoupled by using the second one to eliminate $\partial \Omega_2 / \partial t$ in terms of Ω_1:

$$\frac{\partial \Omega_2}{\partial t} = \omega \, \Omega_1 \,, \tag{4.3.22}$$

where

$$\omega := \left(\frac{\mathcal{I}_3 - \mathcal{I}_1}{\mathcal{I}_1} \right) \Omega_3 \,. \tag{4.3.23}$$

Plugging (4.3.22) into the time derivative of the first of equations (4.3.20) gives an evolution equation involving Ω_1 alone:

$$\frac{\partial^2 \Omega_1}{\partial t^2} + \omega^2 \Omega_1 = 0 \,. \tag{4.3.24}$$

This is the equation of a simple harmonic oscillator and so it has general solution

$$\Omega_1(t) = A \, \cos(\omega t + \delta) \,, \tag{4.3.25}$$

where A and δ are integration constants. Using this in (4.3.22) then gives

$$\Omega_2(t) = A \, \sin(\omega t + \delta) \,, \tag{4.3.26}$$

where a third integration constant must be chosen to vanish in order to satisfy the first of eqs. (4.3.20).

Combining all components the body-frame form of the rotation vector is

$$\boldsymbol{\Omega} = A \Big[\cos(\omega t) \, \hat{\mathbf{e}}_1 + \sin(\omega t) \, \hat{\mathbf{e}}_2 \Big] + \Omega_3 \, \hat{\mathbf{e}}_3 \,, \tag{4.3.27}$$

where we choose our basis so that $\hat{\mathbf{e}}_2$ is orthogonal to $\boldsymbol{\Omega}$ at the initial time (which we choose to be at $t_0 = 0$). Comparing to earlier sections then shows that $A^2 = \Omega_1^2 + \Omega_2^2 = \Omega_\perp^2$. The corresponding angular momentum is then

$$\mathbf{M} = \Omega_1 \mathcal{I}_1 \Big[\cos(\omega t) \, \hat{\mathbf{e}}_1 + \sin(\omega t) \, \hat{\mathbf{e}}_2 \Big] + \mathcal{I}_3 \Omega_3 \, \hat{\mathbf{e}}_3 \,, \tag{4.3.28}$$

and so

$$\mathbf{M} \cdot \boldsymbol{\Omega} = \mathcal{I}_1 \Omega_1^2 + \mathcal{I}_3 \Omega_3^2 \quad \text{and} \quad \mathbf{M}^2 = \mathcal{I}_1^2 \Omega_1^2 + \mathcal{I}_3^2 \Omega_3^2 \,, \tag{4.3.29}$$

and (compare with (4.3.12) and (4.3.13))

$$\mathbf{\Omega} = \frac{\mathbf{M}}{\mathcal{I}_1} + \left(\frac{\mathcal{I}_1 - \mathcal{I}_3}{\mathcal{I}_1}\right) \Omega_3 \,\hat{\mathbf{e}}_3 = \frac{\mathbf{M}}{\mathcal{I}_1} - \omega \,\hat{\mathbf{e}}_3 \,. \tag{4.3.30}$$

This is the body-frame version of what we found for the inertial frame in §4.3.1 (with somewhat more effort). In the inertial frame we saw that **M** is fixed and $\mathbf{\Omega}$ precesses around it with frequency ω. The body's symmetry axis also precesses about **M** so that all three directions remain coplanar for all times. In the body frame the principal directions define the natural basis vectors and these remain unchanged in time. In particular the symmetry direction is fixed because it is one of the principal axes. Relative to this frame $\mathbf{\Omega}$ precesses about the symmetry axis but in the opposite direction from what was found in the inertial frame. **M** does so as well. This apparent evolution of **M** in the body frame is not inconsistent with conservation of **M** because of eq. (4.3.17). The pictures in the two frames agree.

Asymmetric Top

The equations of motion for an asymmetric top can be approached in a similar way, starting from (4.3.18). Although the solution in the general case is more complicated in this section we start by exploring evolution that is close to a simple case: rotation along one of the principal axes. A motivation for doing so is to study the stability of these simplest solutions.

We start with rotations that are close to but not exactly about the symmetry axis $\mathbf{e}_3$. To this end we write $\Omega_3 = \Omega + \omega_3(t)$, $\Omega_2 = \omega_2(t)$ and $\Omega_1 = \omega_1(t)$ where the ω_i's are all much smaller than the constant value Ω. Because they are small we Taylor expand eqs. (4.3.18) in powers of ω_i and keep only the leading (linear) term. This leads to the following approximate evolution equations:

$$\mathcal{I}_1 \frac{\partial \omega_1}{\partial t} + (\mathcal{I}_3 - \mathcal{I}_2)\omega_2 \Omega \simeq 0 \,, \quad \mathcal{I}_2 \frac{\partial \omega_2}{\partial t} + (\mathcal{I}_1 - \mathcal{I}_3)\omega_1 \Omega \simeq 0 \,, \tag{4.3.31}$$

together with $\mathcal{I}_3 \, \partial\omega_3/\partial t \simeq 0$. The last of these implies ω_3 is a constant which can be absorbed into Ω (which allows us to take $\omega_3 = 0$).

Differentiating the first equation and using the second one to eliminate $\partial\omega_2/\partial t$ (or using the same type of argument to derive an equation involving only ω_2) then shows that $y = \omega_1(t)$ and $y = \omega_2(t)$ must both solve the equation

$$\frac{\partial^2 y}{\partial t^2} + \nu y = 0 \quad \text{where} \quad \nu := \left[\frac{(\mathcal{I}_3 - \mathcal{I}_2)(\mathcal{I}_3 - \mathcal{I}_1)}{\mathcal{I}_1 \mathcal{I}_2}\right] \Omega^2 \,. \tag{4.3.32}$$

This has oscillatory solutions $y = A \sin(\lambda t + c)$ – with integration constants A and c – if $\nu = \lambda^2 > 0$. But it instead has non-oscillatory exponential solutions $y = Ae^{\lambda t} + Be^{-\lambda t}$ if $\nu = -\lambda^2 < 0$.

The $\nu > 0$ case corresponds to oscillations about a stable solution and the $\nu < 0$ case describes the exponential departure from an unstable solution. We see from this that if $\mathcal{I}_3$ is either the largest or the smallest of the three principal moments then a small perturbation away from constant rotations about the symmetry axis $\hat{\mathbf{e}}_3$ produces an angular velocity $\boldsymbol{\Omega}$ that just precesses about a stable zeroeth order solution. But if $\mathcal{I}_3$ is neither the largest or the smallest eigenvalue then small perturbations start to evolve away from the unstable initial rotation. Repeating the above arguments for perturbations about an initial rotation about the $\hat{\mathbf{e}}_1$ or $\hat{\mathbf{e}}_2$ directions leads to the same conclusion: rotation about the principal axis for the largest or smallest principal moment is stable but rotations about the middle-sized principal moment is unstable. It is the existence of instability that suggests that evolution in general can be fairly complicated.

4.3.3 The Poinsot construction

Returning to the full evolution equations, (4.3.18) provide a system of nonlinear differential equations whose solution describes general rigid-body motion in the absence of applied forces (or in the presence of forces if these apply no net torque). Although nonlinear these equations admit several integrations because the energy and angular momentum are both conserved, and this allows the problem of the free top to be reduced to quadratures, much as was the case for the two-body problem of §1.2.1 with a central force.

The integrals involved are elliptic integrals and are not that illuminating about the physics of what is going on, but there is a geometrical formulation – the *Poinsot construction* – that makes some features of the motion easier to visualize. The starting point for this line of thought is to interpret geometrically what the conservation laws for kinetic energy, K, and for the magnitude of angular momentum, $\mathcal{M} = |\mathbf{M}|$, express:

$$2K = \mathcal{I}_1\Omega_1^2 + \mathcal{I}_2\Omega_2^2 + \mathcal{I}_3\Omega_3^2 \quad \text{and} \quad \mathcal{M}^2 = \mathcal{I}_1^2\Omega_1^2 + \mathcal{I}_2^2\Omega_2^2 + \mathcal{I}_3^2\Omega_3^2 \,. \tag{4.3.33}$$

Both of these can be regarded as the equation of an ellipsoid in the three-dimensional space whose coordinates are the Ω_i's, as may be seen by writing them as

$$\left(\frac{\mathcal{I}_1}{2K}\right)\Omega_1^2 + \left(\frac{\mathcal{I}_2}{2K}\right)\Omega_2^2 + \left(\frac{\mathcal{I}_3}{2K}\right)\Omega_3^2 = 1 \,, \tag{4.3.34}$$

called the *inertia ellipsoid*, and

$$\left(\frac{\mathcal{I}_1^2}{\mathcal{M}}\right)\Omega_1^2 + \left(\frac{\mathcal{I}_2^2}{\mathcal{M}}\right)\Omega_2^2 + \left(\frac{\mathcal{I}_3^2}{\mathcal{M}}\right)\Omega_3^2 = 1 \,. \tag{4.3.35}$$

The point of this observation is this: the evolution of $\Omega_i(t)$ can be thought of as being inscribed within two ellipsoids, and so the motion that is traced out must lie on their intersection. It is convenient to label the principal axes so that $\mathcal{I}_1 > \mathcal{I}_2 > \mathcal{I}_3$, in which case Ω_3 is in the direction of the longest axis of both ellipsoids and Ω_1 is in the direction of the smallest for both. Fig. 19 shows what the curves of intersection look like when drawn on the inertia ellipsoid for various values of $\mathcal{M}$.

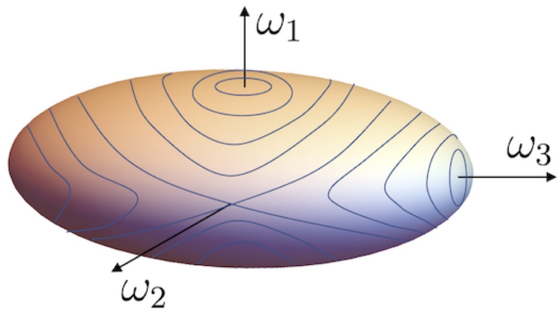

Figure 19. Illustration of the intersection of the two ellipsoids described by eqs. (4.3.33). Figure taken from *Classical Dynamics* (lecture notes by David Tong, found here).

What is noteworthy is the resulting trajectories – called *polhode curves* – trace closed curves on the inertia ellipsoid and as a result the motion is periodic (as opposed to chaotic). For motion very close to one of the principal directions the polhode curves are small circles if they are close to either the largest or smallest axes of the ellipsoid (*i.e.* for the Ω_1 and Ω_3 directions), corresponding to the oscillatory solutions found perturbatively in (4.3.32) when $\nu > 0$. But when $\boldsymbol{\Omega}$ points almost along the Ω_2 axis they are crossed lines, corresponding to the unstable solutions to (4.3.32) found above when $\nu < 0$.

4.4 Euler enters the chat (again)

To this point we have had much to say about the angular velocities appearing in rigid-body motion but have had almost nothing to say about how to describe angular positions. This section rectifies that, providing an explicit parameterization of an arbitrary finite rotation R_{ij}. Having explicit position coordinates is useful because it opens up the ability to use Lagrangian methods as is useful when treating rigid bodies subject to nonzero external forces and torques.

4.4.1 Euler's theorem

Before constructing a general rotation there is a side-issue to resolve that might otherwise be bothersome: we know from explicit construction that rotations are described by orthogonal matrices, since (for example) a rotation about the x, y or z axis by an angle θ has the form

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \quad R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad \text{or} \quad R_z = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.4.1}$$

each of which is easily seen to be orthogonal. But how do we know that an orthogonal matrix must be a rotation?

Strictly speaking they do not, since taking the determinant of $RR^T = I$ shows that $\det R = \pm 1$. But rotations are continously connected to the identity transformation (which has determinant 1) and so rotations all must have determinant equal to unity. But any orthogonal transformation with determinant -1 – what are called *improper* transformations – can be obtained from those with determinant $+1$ - not surprisingly called *proper* transformations – by multiplying by a specific matrix

$$\mathbb{P} := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4.4.2}$$

which implements a reflection in all three spatial directions (a transformation called *parity*). So the more precise question is whether or not all 3×3 orthogonal matrices with unit determinant are rotations about some axis. Orthogonal transformations with unit determinant make up what is called the group of 'special' orthogonal matrices – or $SO(3)$ for short – to distinguish them from the generic $O(3)$ group of 3×3 orthogonal matrices.

Euler's theorem provides the answer: they are. This seems plausible because orthogonality, $R^T R = I$, provides a symmetric matrix's worth of conditions on the 9 elements of a generic real 3×3 matrix. Since a symmetric 3×3 matrix has six independent elements this suggests $SO(3)$ is a $9 - 6 = 3$ parameter set of matrices, in agreement with the freedom to rotate about each of three independent directions shown in (4.4.1).

But there is more to the argument than just counting parameters. Any rotation can be written in the form (4.4.1) if we adapt the coordinate directions so that one of them lies along the axis of rotation. We must show that any element of $SO(3)$ can be written in this way. This will always be possible if and only if the eigenvalues[12] of the matrix are $\{1, e^{i\theta}, e^{-i\theta}\}$, so we must think through the implications of the eigenvalue condition

$$R\mathbf{u} = \lambda \mathbf{u}\,, \tag{4.4.3}$$

for orthogonal matrices.

The first thing to observe is that any eigenvalue of a unitary matrix must satisfy $\lambda = e^{i\alpha}$ for some real α. This follows because $U\mathbf{u} = \lambda\mathbf{u}$ implies

$$\mathbf{u}^\dagger \mathbf{u} = \mathbf{u}^\dagger U^\dagger U \mathbf{u} = |\lambda|^2 \mathbf{u}^\dagger \mathbf{u} \qquad (\text{whenever} U^\dagger U = I)\,, \tag{4.4.4}$$

[12]A square matrix M can always be diagonalized by a unitary transformation so long as it commutes with its adjoint: $M^\dagger M = M M^\dagger$ (what is called a *normal* matrix), so in particular any real orthogonal matrix can be diagonalized in this way. There is no guarantee the transformation that does so must be real and so also there is no guarantee the eigenvalues must be real.

and so $|\lambda| = 1$. This result is also true in particular for orthogonal matrices, since these are unitary. It also follows that any real eigenvalue of a unitary matrix must be ± 1.

The second thing to observe is that any eigenvalue λ of R must satisfy the condition $\det(R - \lambda I) = 0$, which for 3×3 matrices is a cubic polynomial in λ with real coefficients:

$$c_1 \lambda^3 + c_2 \lambda^2 + c_3 \lambda + c_4 = 0 \,. \tag{4.4.5}$$

But because the coefficients c_i are real it follows that if λ is an eigenvalue then its complex conjugate λ^* must also be an eigenvalue. This means that at least one of the three eigenvalues must be real. This same conclusion also follows because the left-hand side of (4.4.5) necessarily changes sign at least once as λ is taken from $-\infty$ to $+\infty$ through real values, since for large $|\lambda|$ its sign is dominated by the cubic term alone. We use the fact that at least one eigenvalue is real together with the freedom to relabel the eigenvalues to ensure $\lambda_3 = \lambda_3^*$. It might happen that a second eigenvalue is also real, but if it is then they all must be real.

The third thing to observe is that the product of eigenvalues is the determinant and so the condition $\det R = 1$ implies $\lambda_1 \lambda_2 \lambda_3 = 1$. First suppose all three eigenvalues are real. In this case they must all be either $+1$ or -1. Since their product is $+1$ they are either all equal $\lambda_1 = \lambda_2 = \lambda_3 = +1$ (in which case R is the unit matrix), or two of them are -1 and one is $+1$. Either case is a special instance of $\{1, e^{i\theta}, e^{-i\theta}\}$, either with $\theta = 0$ or $\theta = \pi$. Alternatively, if one eigenvalue is complex – say $\lambda_1 = e^{i\alpha}$ – then we must have $\lambda_2 = e^{-i\alpha}$ for the same α and so $\lambda_1 \lambda_2 \lambda_3 = \lambda_3 = 1$. This also agrees with $\{1, e^{i\theta}, e^{-i\theta}\}$, with $\theta = \alpha$.

These arguments show that any $SO(3)$ matrix – *i.e.* those for which $R^T R = I$ and $\det R = 1$ – is necessarily a rotation about some axis.

4.4.2 Euler Angles

It is useful to have an explicit convention for how to label an arbitrary 3×3 rotation, and *Euler angles* provide a convenient way to do so.[13] We choose the following product of the matrices in (4.4.1):

$$\begin{aligned} R(\theta, \phi, \psi) &= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi & \sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \cos\theta\sin\phi\cos\psi & -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} . \end{aligned} \tag{4.4.6}$$

If the rotation takes (*e.g.* the inertial basis) $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to (*e.g.* the body frame basis) $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, then the above convention his corresponds to doing so in three steps. First we

[13] Beware that not all definitions of Euler angles in the literature are exactly the same (though they are equivalent to one another). So be sure of your conventions before lifting any formulae from books!

rotate through an angle ϕ about the initial $\mathbf{e}_z$ direction, arriving at an intermediate basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3 = \mathbf{e}_z\}$. Next we rotate through an angle θ about the *new* axis $\mathbf{e}'_1$, arriving at a second intermediate basis $\{\mathbf{e}''_1 = \mathbf{e}'_1, \mathbf{e}''_2, \mathbf{e}''_3\}$. Finally we rotate about the axis $\mathbf{e}''_3$ in this newest basis to reach the final frame $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3 = \mathbf{e}''_3\}$. The last rotation does not undo the first rotation because they are around different 'z' axes.

The idea is that any 3-dimensional rotation can be decomposed in this way, so we can use the angles $\{\theta, \phi, \psi\}$ as a set of coordinates describing an arbitrary 3-dimensional rotation. In particular we can use these angles as a function of time $\{\theta(t), \phi(t), \psi(t)\}$ as a way to label the orientation at any instant of a rigid body relative to a fixed frame that instantaneously moves with the same position and velocity as the rigid body's centre of mass at that instant.

In terms of Euler angles the body frame is given by $\hat{\mathbf{e}}_i = R_{ij}\, \mathbf{e}_j$, which using (4.4.6) becomes

$$
\begin{aligned}
\hat{\mathbf{e}}_1 &= (\cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi)\,\mathbf{e}_x + (\sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi)\,\mathbf{e}_y + \sin\theta\sin\psi\,\mathbf{e}_z \\
\hat{\mathbf{e}}_2 &= -(\cos\phi\sin\psi + \cos\theta\sin\phi\cos\psi)\,\mathbf{e}_x + (-\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi)\,\mathbf{e}_y + \sin\theta\cos\psi\,\mathbf{e}_z \\
\hat{\mathbf{e}}_3 &= \sin\theta\sin\phi\,\mathbf{e}_x - \sin\theta\cos\phi\,\mathbf{e}_y + \cos\theta\,\mathbf{e}_z\,.
\end{aligned} \tag{4.4.7}
$$

Inverting gives the inertial basis vectors in terms of the body frame: $\mathbf{e}_i = R_{ji}\,\hat{\mathbf{e}}_j$ and so

$$
\begin{aligned}
\mathbf{e}_x &= -(\cos\phi\sin\psi + \cos\theta\sin\phi\cos\psi)\,\hat{\mathbf{e}}_1 + (\cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi)\,\hat{\mathbf{e}}_2 + \sin\theta\sin\phi\,\hat{\mathbf{e}}_3 \\
\mathbf{e}_y &= (\sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi)\,\hat{\mathbf{e}}_1 + (-\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi)\,\hat{\mathbf{e}}_2 - \sin\theta\cos\psi\,\hat{\mathbf{e}}_3 \\
\mathbf{e}_z &= \sin\theta\sin\psi\,\hat{\mathbf{e}}_1 + \sin\theta\cos\psi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3\,.
\end{aligned} \tag{4.4.8}
$$

In order to make contact with the previous sections we must obtain an expression for the components of the body's angular velocity $\boldsymbol{\Omega}$ in terms of the derivatives $\dot{\theta}$, $\dot{\phi}$ and $\dot{\psi}$. This is not done simply by differentiating $R(t)$ since the vector $\boldsymbol{\Omega}$ is defined in terms of the matrix A when we write $R = I + A$ for rotations close to the identity transformation. To determine how it is defined we go back to the definition of how $R(t)$ is related to the motion of a point P with position vector $\mathbf{r}(t)$ somewhere on the rigid body. This can be expanded in either inertial or body frames, with

$$\mathbf{r}(t) = x(t)\,\mathbf{e}_x + y(t)\,\mathbf{e}_y + z(t)\,\mathbf{e}_z \quad \text{(inertial frame)} \tag{4.4.9a}$$

$$= \hat{x}_1\,\hat{\mathbf{e}}_1(t) + \hat{x}_2\,\hat{\mathbf{e}}_2(t) + \hat{x}_3\,\hat{\mathbf{e}}_3(t) \quad \text{(body frame)}\,, \tag{4.4.9b}$$

where $\hat{x}_1$, $\hat{x}_2$ and $\hat{x}_3$ are the coordinate distances from P to the centre of mass (and so are time-independent, though the basis vectors $\hat{\mathbf{e}}_i(t)$ depend on time) while $x(t)$, $y(t)$ and $z(t)$ describe the position of P relative to a time-independent inertial frame $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

Writing the body-frame axis relative to the inertial frame as

$$\hat{\mathbf{e}}_i(t) = R_{ij}(t)\,\mathbf{e}_j\,, \tag{4.4.10}$$

which also implies $\mathbf{e}_i = R_{ji}(t)\,\hat{\mathbf{e}}_j$, we see that

$$\dot{\mathbf{r}}(t) = \hat{x}_i \dot{\hat{\mathbf{e}}}_i = \hat{x}_i \dot{R}_{ij}\,\mathbf{e}_j = \hat{x}_i \dot{R}_{ij}\,R_{kj}\,\hat{\mathbf{e}}_k = \hat{x}_i (\dot{R}\,R^T)_{ik}\,\hat{\mathbf{e}}_k\,. \tag{4.4.11}$$

This is to be compared with the expression (3.2.5) for the velocity vector for a point moving in the body frame

$$\dot{\mathbf{r}} = \boldsymbol{\Omega} \times \mathbf{r} = \epsilon_{kji}\,\Omega_j \hat{x}_i\,\hat{\mathbf{e}}_k \tag{4.4.12}$$

leading to the expression $\epsilon_{ikj}\,\Omega_j = (\dot{R}R^T)_{ik}$ and so

$$\Omega_i = \tfrac{1}{2}\epsilon_{ijk}\,(\dot{R}R^T)_{jk}\,, \tag{4.4.13}$$

where $(\dot{R}R^T)_{jk} = -(\dot{R}R^T)_{kj}$ follows[14] from differentiating the condition $RR^T = I$.

The hard way to compute $\boldsymbol{\Omega}$ is to explicitly differentiate R and then use the result in (4.4.13). Simpler is to use that $\boldsymbol{\Omega}$ points in the direction of the axis about which the rotation takes place and has magnitude given by the amount of angular change, and to recognize that the $\boldsymbol{\Omega}$'s for successive rotations simply add to one another if the rotations are infinitesimal. It then follows from the definitions of the Euler angles that

$$\boldsymbol{\Omega} = \dot{\phi}\,\mathbf{e}_z + \dot{\theta}\,\mathbf{e}_1' + \dot{\psi}\,\hat{\mathbf{e}}_3\,, \tag{4.4.14}$$

where the vectors $\mathbf{e}_z$ and $\mathbf{e}_1'$ are related to the body-frame axes $\hat{\mathbf{e}}_i$ by

$$\mathbf{e}_1' = \cos\psi\,\hat{\mathbf{e}}_1 - \sin\psi\,\hat{\mathbf{e}}_2 \quad \text{and} \quad \mathbf{e}_z = \sin\theta\sin\psi\,\hat{\mathbf{e}}_1 + \sin\theta\cos\psi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3\,. \tag{4.4.15}$$

Combining everything gives the desired expression for $\boldsymbol{\Omega}$ in the body frame:

$$\boldsymbol{\Omega} = (\dot{\theta}\,\cos\psi + \dot{\phi}\,\sin\theta\sin\psi)\hat{\mathbf{e}}_1 + (\dot{\phi}\,\sin\theta\cos\psi - \dot{\theta}\,\sin\psi)\,\hat{\mathbf{e}}_2 + (\dot{\psi} + \dot{\phi}\,\cos\theta)\,\hat{\mathbf{e}}_3\,. \tag{4.4.16}$$

An identical argument gives $\boldsymbol{\Omega}$ in the inertial frame, which replaces eqs. (4.4.15) with

$$\mathbf{e}_1' = \cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y \quad \text{and} \quad \hat{\mathbf{e}}_3 = \sin\theta\sin\phi\,\mathbf{e}_x - \sin\theta\cos\phi\,\mathbf{e}_y + \cos\theta\,\mathbf{e}_z\,, \tag{4.4.17}$$

and so the result for the angular momentum in the inertial frame is

$$\boldsymbol{\Omega} = (\dot{\theta}\,\cos\phi + \dot{\psi}\,\sin\theta\sin\phi)\mathbf{e}_x + (-\dot{\psi}\,\sin\theta\cos\phi + \dot{\theta}\,\sin\phi)\,\mathbf{e}_y + (\dot{\phi} + \dot{\psi}\,\cos\theta)\,\mathbf{e}_z\,. \tag{4.4.18}$$

Now that we have generalized coordinates we can write down the lagrangian describing rigid body dynamics. The simplest starting point when doing so is an isolated system for which there are no net forces or torques. When doing so it is most convenient to adapt the body frame to align with the body's principal axes. In this case eqs. (4.4.7) or (4.4.8) show that $\mathbf{e}_z \cdot \hat{\mathbf{e}}_3 = \cos\theta$ and so θ gives the direction of the body frame's $\mathcal{I}_3$ axis relative to $\mathbf{e}_z$.

[14]Explicitly: $0 = (\mathrm{d}/\mathrm{d}t)(RR^T) = \dot{R}R^T + R\dot{R}^T = \dot{R}R^T + (\dot{R}R^T)^T$.

4.4.3 Lagrangian for an isolated rigid body

In the absence of applied forces and torques the Lagrangian is just the kinetic energy, $L = K = \frac{1}{2} I_{ij}\, \Omega_i \Omega_j$, and so for coordinates adapted to the principal axes this becomes

$$\begin{aligned} L &= \tfrac{1}{2}\Big(\mathcal{I}_1\, \Omega_1^2 + \mathcal{I}_2\, \Omega_2^2 + \mathcal{I}_3\, \Omega_3^2\Big) \qquad (4.4.19)\\ &= \tfrac{1}{2}\Big[\mathcal{I}_1(\dot\theta\, \cos\psi + \dot\phi\, \sin\theta \sin\psi)^2 + \mathcal{I}_2(\dot\phi\, \sin\theta\cos\psi - \dot\theta\, \sin\psi)^2 + \mathcal{I}_3(\dot\psi + \dot\phi\, \cos\theta)^2\Big] . \end{aligned}$$

The three generalized momenta are $p_A = \partial L/\partial \dot q^A$, which for this Lagrangian becomes

$$\begin{aligned} p_\theta &= \mathcal{I}_1(\dot\theta\, \cos\psi + \dot\phi\, \sin\theta\sin\psi)\cos\psi - \mathcal{I}_2(\dot\phi\, \sin\theta\cos\psi - \dot\theta\, \sin\psi)\sin\psi \\ &= \mathcal{I}_1\Omega_1 \cos\psi - \mathcal{I}_2\Omega_2 \sin\psi \qquad (4.4.20a)\\ p_\phi &= \mathcal{I}_1(\dot\theta\, \cos\psi + \dot\phi\, \sin\theta\sin\psi)\sin\theta\sin\psi + \mathcal{I}_2(\dot\phi\, \sin\theta\cos\psi - \dot\theta\, \sin\psi)\sin\theta\cos\psi \\ &\qquad + \mathcal{I}_3(\dot\psi + \dot\phi\, \cos\theta)\cos\theta \\ &= \mathcal{I}_1\Omega_1 \sin\theta\sin\psi + \mathcal{I}_2\Omega_2\sin\theta\cos\psi + \mathcal{I}_3\Omega_3\cos\theta \qquad (4.4.20b)\\ p_\psi &= \mathcal{I}_3(\dot\psi + \dot\phi\, \cos\theta) = \mathcal{I}_3\Omega_3\,, \qquad (4.4.20c) \end{aligned}$$

so the Euler-Lagrange equations $\dot p_A = \partial L/\partial q^A$ then read

$$\begin{aligned} \dot p_\theta &= \mathcal{I}_1(\dot\theta\, \cos\psi + \dot\phi\, \sin\theta\sin\psi)\dot\phi\, \cos\theta\sin\psi \qquad (4.4.21)\\ &\qquad + \mathcal{I}_2(\dot\phi\, \sin\theta\cos\psi - \dot\theta\, \sin\psi)\dot\phi\, \cos\theta\cos\psi - \mathcal{I}_3(\dot\psi + \dot\phi\, \cos\theta)\dot\phi\, \sin\theta\,, \end{aligned}$$

$$\dot p_\psi = (\mathcal{I}_1 - \mathcal{I}_2)(\dot\phi\, \sin\theta\cos\psi - \dot\theta\, \sin\psi)(\dot\phi\, \sin\theta\sin\psi + \dot\theta\, \cos\psi)\,, \qquad (4.4.22)$$

and

$$\dot p_\phi = 0\,. \qquad (4.4.23)$$

These equations show that ϕ is generically an ignorable coordinate, inasmuch as its equation of motion is a conservation law: p_ϕ is time-independent. The corresponding symmetry is $\phi \to \phi +$ constant, which (4.4.6) shows amounts to a rotation about the $\mathbf{e}_z$ axis. The general arguments of §2.3 show that this corresponds to the component

$$p_\phi = M_z = \mathbf{e}_z \cdot \mathbf{M} \qquad (4.4.24)$$

of angular momentum.

Since we know angular momentum $\mathbf{M}$ is conserved for an isolated top when measured in an inertial frame there is no loss of generality in adapting our inertial frame basis vectors so that $\mathbf{e}_z$ points in the same direction as does $\mathbf{M}$. With this choice p_ϕ can be interpreted as the total angular momentum, $M = |\mathbf{M}|$ and then (4.4.17) shows that θ can be interpreted as the angle between $\mathbf{M}$ and the body axis $\hat{\mathbf{e}}_3$.

4.4.4 Symmetric top revisited

Eq. (4.4.22) also becomes a conservation law when $\mathcal{I}_1 = \mathcal{I}_2$, the case of a symmetric top. This section specializes this example in more detail since results can then be compared with the discussion in §4.3.1 and §4.3.2.

In the special case $\mathcal{I}_1 = \mathcal{I}_2$ the Lagrangian (4.4.19) reduces to

$$L = \tfrac{1}{2}\Big[\mathcal{I}_1(\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \mathcal{I}_3(\dot\psi + \dot\phi \cos\theta)^2\Big], \tag{4.4.25}$$

and so the three generalized momenta of (4.4.20) simplify to

$$p_\theta = \mathcal{I}_1\,\dot\theta\,, \quad p_\phi = \mathcal{I}_1\,\dot\phi\,\sin^2\theta + \mathcal{I}_3(\dot\psi + \dot\phi\,\cos\theta)\cos\theta \tag{4.4.26}$$

and

$$p_\psi = \mathcal{I}_3(\dot\psi + \dot\phi\,\cos\theta) = \mathcal{I}_3\Omega_3\,, \tag{4.4.27}$$

with Euler-Lagrange equations

$$\dot p_\theta = (\mathcal{I}_1 - \mathcal{I}_3)\dot\phi^2\,\sin\theta\cos\theta - \mathcal{I}_3\dot\psi\dot\phi\,\sin\theta \quad \text{and} \quad \dot p_\psi = \dot p_\phi = 0\,. \tag{4.4.28}$$

In this case both ϕ and ψ are ignorable coordinates, whose equations of motion express conservation laws. As mentioned earlier, shifts of ϕ correspond to rotations about the $\mathbf{e}_z$ axis, leading to the identification of p_ϕ with the component of angular momentum in the $\mathbf{e}_z$ direction in the inertial frame, and so $p_\phi = \mathcal{M} = |\mathbf{M}|$ if we choose $\mathbf{e}_z$ to point in the same direction as $\mathbf{M}$.

Shifts of ψ, on the other hand, correspond to rotations about the body-frame $\hat{\mathbf{e}}_3$ axis, as can be seen by inspecting (4.4.6), with conservation of p_ψ corresponding to the conservation of the component $\widehat{M}_3 = \hat{\mathbf{e}}_3 \cdot \mathbf{M}$ of angular momentum obtained by projecting onto the $\hat{\mathbf{e}}_3$ axis. In §4.3.1 this conservation was used to conclude that the angular velocity, Ω_3, about the symmetry direction $\hat{\mathbf{e}}_3$ must be constant, and eq. (4.4.27) shows that this conclusion also follows from $\dot p_\psi = 0$ in the Lagrangian formulation.

Since both $|\mathbf{M}|$ and $\hat{e}_3 \cdot \mathbf{M}$ are constants it follows that the angle between $\mathbf{M}$ and $\hat{\mathbf{e}}_3$ also does not depend on time. But in the case where we choose $\mathbf{e}_z$ parallel to $\mathbf{M}$ we know – see the discussion below eq. (4.4.24) – that the coordinate θ is the angle between $\mathbf{M}$ and $\hat{\mathbf{e}}_3$ and so it must be true that in this frame θ is time-independent: $\dot\theta = 0$. Eqs. (4.4.26) and (4.4.27) then show that $\dot\phi$ and $\dot\psi$ are also time-independent.

Finally, $\dot\theta = 0$ implies $p_\theta = 0$ and setting $\dot p_\theta = 0$ in the evolution equation (4.4.28) gives

$$\Big[(\mathcal{I}_1 - \mathcal{I}_3)\dot\phi\,\cos\theta - \mathcal{I}_3\dot\psi\Big]\dot\phi\,\sin\theta = 0\,, \tag{4.4.29}$$

from which we conclude (assuming $\dot\phi\sin\theta \neq 0$)

$$\dot\psi = \left(\frac{\mathcal{I}_1 - \mathcal{I}_3}{\mathcal{I}_3}\right)\dot\phi\,\cos\theta\,. \tag{4.4.30}$$

Using this and $\dot{\theta} = 0$ in expression (4.4.16) then implies the body-frame angular velocity is

$$\boldsymbol{\Omega} = \dot{\phi} \left[\sin\theta \sin\psi \, \hat{\mathbf{e}}_1 + \sin\theta \cos\psi \, \hat{\mathbf{e}}_2 + \left(\frac{\mathcal{I}_1}{\mathcal{I}_3} \right) \cos\theta \, \hat{\mathbf{e}}_3 \right] . \tag{4.4.31}$$

The angular momentum therefore is

$$\mathbf{M} = \mathcal{I}_1 \dot{\phi} \Big(\sin\theta \sin\psi \, \hat{\mathbf{e}}_1 + \sin\theta \cos\psi \, \hat{\mathbf{e}}_2 + \cos\theta \, \hat{\mathbf{e}}_3 \Big) = \mathcal{I}_1 \dot{\phi} \, \mathbf{e}_z \,, \tag{4.4.32}$$

where the last equality uses (4.4.8). This verifies that $\mathbf{M}$ is a constant vector pointing in the $\mathbf{e}_z$ direction.

Eq. (4.4.31) describes precession within the body frame – precisely as found in §4.3.2 – because integrating (4.4.30) shows that $\boldsymbol{\Omega}$ is time-dependent due to its dependence on $\psi(t) = \psi_0 + \Omega_{\rm pr} t$, where the precession velocity is

$$\Omega_{\rm pr} = \dot{\psi} = \left(\frac{\mathcal{I}_1 - \mathcal{I}_3}{\mathcal{I}_3} \right) \dot{\phi} \, \cos\theta = \left(\frac{\mathcal{I}_1 - \mathcal{I}_3}{\mathcal{I}_1} \right) \Omega_3 \,, \tag{4.4.33}$$

in agreement with (4.3.13). The last equality uses (4.4.31) to trade $\dot{\phi}$ for Ω_3.

The behaviour of $\boldsymbol{\Omega}$ in the inertial frame is found by converting the basis vectors using (4.4.8) and (4.4.7), leading to

$$\begin{aligned} \boldsymbol{\Omega} &= \dot{\phi} \left[\mathbf{e}_z + \left(\frac{\mathcal{I}_1 - \mathcal{I}_3}{\mathcal{I}_3} \right) \cos\theta \, \hat{\mathbf{e}}_3 \right] = \dot{\phi} \, \mathbf{e}_z + \Omega_{\rm pr} \, \hat{\mathbf{e}}_3 \\ &= \Omega_{\rm pr} \sin\theta \Big(\sin\phi \, \mathbf{e}_x - \cos\phi \, \mathbf{e}_y \Big) + \Big(\dot{\phi} + \Omega_{\rm pr} \cos\theta \Big) \, \mathbf{e}_z \,, \end{aligned} \tag{4.4.34}$$

In this frame the time-dependence of $\boldsymbol{\Omega}$ arises through the dependence on $\phi(t)$, and (4.4.32) shows that the precession rate in the inertial frame is $\dot{\phi} = \mathcal{M}/\mathcal{I}_1$ (compare with (4.3.15)).

4.5 Dynamics with Applied Forces

We next ask about evolution in the presence of external applied forces, in which case the equations of motion become (see (1.3.3) and (1.4.9) and we assume rotation invariance so that the net torque due to internal forces vanishes):

$$\dot{\mathbf{P}} = \mathbf{F} \quad \text{and} \quad \dot{\mathbf{M}} = \boldsymbol{\tau} \,, \tag{4.5.1}$$

where $\mathbf{P} = M\dot{\mathbf{R}}$ is the body's total momentum and $\mathbf{M}$ is its angular momentum about the centre of mass as defined in (4.1.37), while

$$\mathbf{F} = \sum_a \mathbf{F}_a^{\rm ext} \quad \text{and} \quad \boldsymbol{\tau} = \sum_a \mathbf{s}_a \times \mathbf{F}_a^{\rm ext} \tag{4.5.2}$$

are the total externally applied forces and the torques to which they give rise. Here $\mathbf{s}_a = \mathbf{r}_a - \mathbf{R}$ so the torque is computed here relative to the position of the centre of mass.

To separate issues we imagine that the net force applied is zero ($\mathbf{F} = 0$) but the net torque is nonzero ($\boldsymbol{\tau} \neq 0$) . In this case having $\mathbf{F} = 0$ ensures that the centre of mass does not accelerate – see for instance (1.3.3) – and so we can (and will) adopt an inertial frame in which the centre of mass does not move. The situation where the net force vanishes but a nonzero net torque exists is called a *couple.*

4.5.1 Slowly varying forces

In general the calculation of the net external torque involves summing over the torque that is applied by external forces atom by atom, but for many practical examples this sum can be simplified. This is true in particular if the force is the result of a field that does not vary appreciably in strength across the size of the rigid body.

Electrostatic force

For instance for electrostatic forces due to an electric field that is approximately constant across the body we have a total force

$$\mathbf{F}^E_a = q_a \, \mathbf{E} \quad \text{and so} \quad \mathbf{F}^E = \sum_a \mathbf{F}^E_a = Q\mathbf{E} \,, \tag{4.5.3}$$

where $Q := \sum_a q_a$ is the body's total charge. The net torque about the centre of mass due to such a force is

$$\boldsymbol{\tau}^E = \sum_a \mathbf{s}_a \times \mathbf{F}^E_a = \mathbf{D}_Q \times \mathbf{E} \,, \tag{4.5.4}$$

where

$$\mathbf{D}_Q = \sum_a q_a \, \mathbf{s}_a \,, \tag{4.5.5}$$

is the net electric dipole carried by the rigid body. Notice that because q_a can have either sign it is possible to have $\mathbf{D}_Q \neq 0$ even if the body itself is electrically neutral so $Q = 0$. Notice also that $\mathbf{D}_Q$ would vanish because of the identity $\sum_a m_a \mathbf{s}_a = 0$ if all of the atoms in a body had the same charge-to-mass ratio (*i.e.* if $\gamma = q_a/m_a$ were the same for all a).

When the body is not electrically neutral (so $Q \neq 0$) we can interpret

$$\mathbf{R}_Q := \frac{\mathbf{D}_Q}{Q} = \frac{1}{Q} \sum_a q_a \, \mathbf{s}_a \tag{4.5.6}$$

as an effective 'centre of charge' for the rigid body. In this case expression (4.5.4) says the net torque applied can be computed 'as if' the total electric force $\mathbf{F}^E = Q\,\mathbf{E}$ were applied at the point $\mathbf{r} = \mathbf{R}_Q$.

Gravitational force

Another approximately constant force often relevant for bodies near the Earth's surface is the force of gravity: $\mathbf{F}_a^{\rm g} = m_a \mathbf{g}$. In this case the net force applied to a rigid body is

$$\mathbf{F}^{\rm g} = \sum_a m_a \mathbf{g} = M\mathbf{g}\,, \tag{4.5.7}$$

where (as usual) $M = \sum_a m_a$ is the total mass. This force can never vanish because $m_a > 0$ for all atoms, but it can easily be balanced by other applied contact forces.

The net torque about the centre of mass on a rigid body produced by gravity then is

$$\boldsymbol{\tau}^{\rm g} = \left(\sum_a m_a \, \mathbf{s}_a \right) \times \mathbf{g} = 0\,, \tag{4.5.8}$$

which vanishes due to the identity $\sum_a m_a \mathbf{s}_a = 0$. More generally, the torque applied by gravity relative to an origin displaced from the centre of mass is

$$\sum_a \mathbf{r}_a \times \mathbf{F}_a^{\rm g} = \sum_a m_a \mathbf{r}_a \times \mathbf{g} = M\mathbf{R} \times \mathbf{g} + \sum_a m_a \mathbf{s}_a \times \mathbf{g} = M\mathbf{R} \times \mathbf{g}\,, \tag{4.5.9}$$

where $\mathbf{r}_a = \mathbf{R} + \mathbf{s}_a$. This shows how the force of gravity acts on the rigid body as if the total force $M\mathbf{g}$ were applied to the centre of mass, and so in particular it does not apply a torque if this torque is taken about the centre of mass.

Magnetic force

The magnetic force acting on each atom due to a magnetic field $\mathbf{B}$ is $\mathbf{F}_a^{\rm m} = q_a \, \dot{\mathbf{r}}_a \times \mathbf{B}$, so when the magnetic field is approximately constant across a rigid body the total magnetic force is given by

$$\mathbf{F}_{\rm tot}^{M} = \sum_a q_a (\dot{\mathbf{r}}_a \times \mathbf{B}) = \left[\sum_a q_a (\dot{\mathbf{R}} + \dot{\mathbf{s}}_a) \right] \times \mathbf{B} = Q\dot{\mathbf{R}} \times \mathbf{B} + \dot{\mathbf{D}}_Q \times \mathbf{B}\,, \tag{4.5.10}$$

where $\mathbf{D}_Q$ is defined in (4.5.5). Again this decomposes into the Lorentz force law for the entire object, using the total charge and centre of mass velocity, plus a contribution from the rate of change of the charge distribution within the centre-of-mass frame.

The net torque applied by a magnetic field similarly is

$$\boldsymbol{\tau}^{M} = \sum_a q_a [\mathbf{r}_a \times (\dot{\mathbf{r}}_a \times \mathbf{B})] = \mathbf{m} \times \mathbf{B}\,, \tag{4.5.11}$$

where

$$\mathbf{m} := \sum_a q_a (\mathbf{r}_a \times \dot{\mathbf{r}}_a)\,. \tag{4.5.12}$$

Notice that $\mathbf{m}$ can be nonzero even if the total charge $Q = \sum_a q_a$ vanishes.

These expressions simplify considerably in the special case that the charges in the body all share the same charge-to-mass ratio: $\gamma := q_a/m_a = Q/M$ is independent of a. In this case the last term in the total force expression (4.5.10) vanishes because

$$\sum_a q_a \mathbf{s}_a = \gamma \sum_a m_a \mathbf{s}_a = 0 \tag{4.5.13}$$

(and the same for its time derivative) by virtue of the definition of the $\mathbf{s}_a$. The net torque also simplies because the identity (4.5.13) implies (4.5.12) becomes

$$\mathbf{m} = \sum_a q_a \Big[(\mathbf{R} + \mathbf{s}_a) \times (\dot{\mathbf{R}} + \dot{\mathbf{s}}_a)\Big] = Q\mathbf{R} \times \dot{\mathbf{R}} + \sum_a q_a (\mathbf{s}_a \times \dot{\mathbf{s}}_a) = \frac{Q}{M}\Big(\mathbf{L} + \mathbf{M}\Big)\,, \tag{4.5.14}$$

where the last equality uses the definitions (4.1.36) and (4.1.37) of 'orbital' and 'intrinsic' angular momentum:

$$\mathbf{L} = M\mathbf{R} \times \dot{\mathbf{R}} \quad \text{and} \quad \mathbf{M} = \sum_a m_a \, \mathbf{s}_a \times \dot{\mathbf{s}}_a \,. \tag{4.5.15}$$

Contact forces

The other kinds of forces commonly encountered for rigid bodies are the contact forces briefly discussed in §1.5. These have their microscopic origins in short range interatomic forces that act between atoms in each of two bodies once these bodies are brought into close enough proximity to one another. For the present purposes what matters is that these forces come with an effective point of application that describes the point to which the total force found by summing over all atoms should be applied if it is to generate the same contribution to the net external force and torque.

4.5.2 When the axis of rotation is fixed

The simplest type of applied force problem for a rigid body assumes there is a fixed axis of rotation about which the body can potentially pivot. This axis is not itself allowed to pivot and so the angular velocity and acceleration are constrained to lie along a fixed axis. The angular behaviour then becomes a one-dimensional problem for which only the magnitude $\Omega(t) = |\mathbf{\Omega}(t)|$ is of interest.

As an illustration of this kind of system consider the example of a compound pendulum.

Worked example: Motion of a compound pendulum

Consider a compound pendulum consisting of a rigid body of mass M and inertia tensor I_{ij} that is constrained to pivot about an axis (labelled A in Fig. 20) and subject to the force of gravity, $\mathbf{F}^{\rm g}$, which we've seen can be regarded as having magnitude Mg and being applied at the object's centre of mass (denoted by C in the figure) that lies a distance ℓ from the axis. What is the reaction force $\mathbf{Q}$ applied at the pivot axis needed to enforce the condition that the body is only free to pivot about the axis A? What is the motion for small displacements from equilibrium? Assume all internal forces are rotation invariant so that all internal torques vanish.

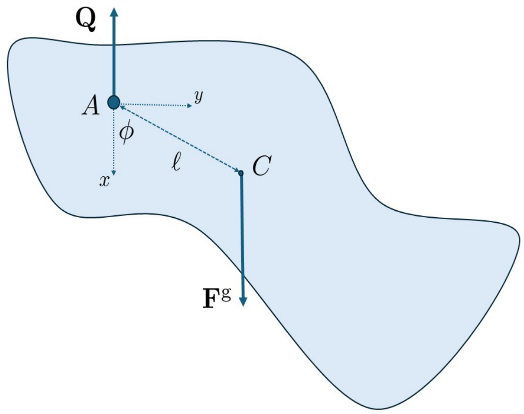

Figure 20. Illustration of the compound pendulum that is free to turns about an axis at which a reaction force $\mathbf{Q}$ keeps the pendulum from doing anything but pivot.

We adopt coordinate directions $\mathbf{e}_x$ and $\mathbf{e}_y$ as indicated by 'x' and 'y' in the figure. With these choices $\mathbf{e}_z = \mathbf{e}_x \times \mathbf{e}_y$ points up out of the page. The force of gravity then is $\mathbf{F}^{\mathrm{g}} = Mg\,\mathbf{e}_x$ and the reaction force $\mathbf{Q}$ is whatever it must be to ensure that the centre of mass moves only in an angular direction about the axis. Writing $\mathbf{R} = \ell(\cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y) =: \ell\,\mathbf{e}_r$ where $\mathbf{e}_r(\phi)$ is the radial unit vector (and only ϕ depends on time) ensures $\dot{\mathbf{R}} = \ell\dot{\phi}\,(-\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y) =: \ell\dot{\phi}\,\mathbf{e}_\phi$, where $\mathbf{e}_\phi = \mathrm{d}\mathbf{e}_r/\mathrm{d}\phi$ satisfies $\mathbf{e}_\phi \cdot \mathbf{e}_r = 0$ and $\mathbf{e}_\phi \cdot \mathbf{e}_\phi = \mathbf{e}_r \cdot \mathbf{e}_r = 1$. Notice that with these definitions $\mathrm{d}\mathbf{e}_\phi/\mathrm{d}\phi = -\cos\phi\,\mathbf{e}_x - \sin\phi\,\mathbf{e}_y = -\mathbf{e}_r$. The centre of mass acceleration then is $\ddot{\mathbf{R}} = \ell\ddot{\phi}\,\mathbf{e}_\phi - \ell\dot{\phi}^2\,\mathbf{e}_r$ and so the equation of motion

$$\dot{\mathbf{P}} = M\ddot{\mathbf{R}} = \mathbf{F}^{\mathrm{g}} + \mathbf{Q} \tag{4.5.16}$$

implies

$$\begin{aligned}\mathbf{Q} &= M\ddot{\mathbf{R}} - \mathbf{F}^{\mathrm{g}} = M\ell\ddot{\phi}(-\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y) - M\ell\dot{\phi}^2(\cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y) - Mg\,\mathbf{e}_x \\ &= -M\Big[\ell(\ddot{\phi}\,\sin\phi + \dot{\phi}^2\,\cos\phi) + g\Big]\mathbf{e}_x + M\ell(\ddot{\phi}\,\cos\phi - \dot{\phi}^2\,\sin\phi)\,\mathbf{e}_y\,.\end{aligned} \tag{4.5.17}$$

At the equilibrium position $\ddot{\mathbf{R}} = 0$ and so $\ddot{\phi} = \dot{\phi} = 0$, in which case $\mathbf{Q}_{\mathrm{eq}} = -Mg\,\mathbf{e}_x$ points directly up.

To describe oscillations about the equilibrium position we require the angular equation

$$\dot{\mathbf{J}} = \boldsymbol{\tau} = \mathbf{R} \times \mathbf{F}^{\mathrm{g}} = \ell(\cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y) \times (Mg\,\mathbf{e}_x) = -Mg\ell\sin\phi\,\mathbf{e}_z\,, \tag{4.5.18}$$

since $\mathbf{Q}$ exerts no torque about the axis. Here $\mathbf{J} = M\mathbf{R} \times \dot{\mathbf{R}} + \mathbf{M}$ is the total angular momentum of the rigid body – see eq. (4.1.35) and (1.4.9) where the internal torques are zero. The motion of the centre of mass contributes

$$\mathbf{L} = M\mathbf{R} \times \dot{\mathbf{R}} = M\ell\mathbf{e}_r \times (\ell\dot{\phi}\,\mathbf{e}_\phi) = M\ell^2\dot{\phi}\,\mathbf{e}_z\,, \tag{4.5.19}$$

which uses $\mathbf{e}_r \times \mathbf{e}_\phi = (\cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y) \times (-\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y) = (\cos^2\phi + \sin^2\phi)(\mathbf{e}_x \times \mathbf{e}_y) = \mathbf{e}_z$.

The angular velocity of the rigid body's rotation about its centre of mass is $\boldsymbol{\Omega} = \dot{\phi}\,\mathbf{e}_z$ since a rotation through an angle $\delta\phi$ about the axis A implies a rotation about the centre of mass by the same angle parallel to the z axis. The internal angular momentum for the rotation about the centre of mass therefore is

$$\mathbf{M} = \mathfrak{I}\,\dot{\phi}\,\mathbf{e}_z \quad \text{where} \quad \mathfrak{I} = I_{ij}n_i n_j = I_{zz} \quad \text{for rotations about the } \mathbf{n} = \mathbf{e}_z \text{ direction}\,. \tag{4.5.20}$$

Combining everything we have $\mathbf{J} = \mathbf{L} + \mathbf{M} = (M\ell^2 + \mathfrak{I})\dot{\phi}\,\mathbf{e}_z$ and so $\dot{\mathbf{J}} = (M\ell^2 + \mathfrak{I})\ddot{\phi}\,\mathbf{e}_z$. Using this in (4.5.18) then implies $\phi(t)$ satisfies

$$(M\ell^2 + \mathfrak{I})\ddot{\phi} = -Mg\ell \sin\phi\,. \tag{4.5.21}$$

An alternative way to derive this equation is to use Lagrangian methods for the variable ϕ. In this case the potential energy due to the gravitational force is $V = -Mg\ell\cos\phi$ and the kinetic energy is (compare to (4.1.5) and (4.1.7))

$$K = \tfrac{1}{2}M\dot{\mathbf{R}}^2 + \tfrac{1}{2}\,I_{ij}\Omega_i\Omega_j = \tfrac{1}{2}\Big(M\ell^2 + \mathfrak{I}\Big)\dot{\phi}^2 \tag{4.5.22}$$

leading to the lagrangian

$$L = K - V = \tfrac{1}{2}\Big(M\ell^2 + \mathfrak{I}\Big)\dot{\phi}^2 + Mg\ell\cos\phi\,, \tag{4.5.23}$$

for which (4.5.21) is the Euler-Lagrange equation $(\mathrm{d}/\mathrm{d}t)(\partial L/\partial\dot{\phi}) = \partial L/\partial\phi$.

For small oscillations we can approximate $\sin\phi \simeq \phi$ in which case (4.5.21) becomes the harmonic oscillator equation

$$\ddot{\phi} = -\omega^2\phi \quad \text{with} \quad \omega^2 = \frac{Mg\ell}{M\ell^2 + \mathfrak{I}}\,, \tag{4.5.24}$$

whose angular frequency ω approaches the simple-pendulum value $\sqrt{g/\ell}$ in the limit $\mathfrak{I} \to 0$.

⋆ ⋆ ⋆

4.5.3 When only one point on the rotation axis is fixed

More challenging is the description of rotational motion in the presence of applied forces when the direction of the rotation axis can change. For simplicity our interest remains the case where the applied forces provide a couple: the net forces balance but the torques do not. For this reason we imagine a reaction force can act on the rigid body, but this time only at a single point on the rotation axis (rather than along its entire length, as was done in the previous section where the axis direction was held fixed).

To this end we describe the motion of a symmetrical top with mass M supported at a point and subject to the torque applied by the gravitational field: the heavy symmetrical top (see Fig. 21). We take the point of support and the centre of mass both to be on the symmetry axis and separated by a distance ℓ. We use a body frame $\hat{\mathbf{e}}_i$ attached to the rigid body and aligned with its principal axes, with $\hat{\mathbf{e}}_3$ corresponding to the principal moment $\mathcal{I}_3$ and the plane perpendicular to $\hat{\mathbf{e}}_3$ corresponding to the two equal principal moments, $\mathcal{I}_1 = \mathcal{I}_2$.

We use Euler angles (ψ, θ, ϕ) to describe the rotation from the inertial frame $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the body frame $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, in the same way as is done in §4.4.4, and so can again use

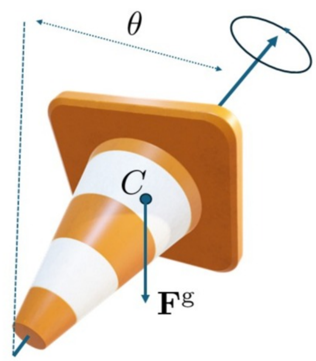

Figure 21. Illustration of the heavy top supported at its base and moving under the influence of the torque due to gravity, drawn here as a force applied to its centre of mass C. The Euler angle θ gives the angle between the symmetry axis of the top and the vertical.

(4.4.7), repeated here for ease of reference:

$$\begin{aligned}
\hat{\mathbf{e}}_1 &= (\cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi)\,\mathbf{e}_x + (\sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi)\,\mathbf{e}_y + \sin\theta\sin\psi\,\mathbf{e}_z \\
\hat{\mathbf{e}}_2 &= -(\cos\phi\sin\psi + \cos\theta\sin\phi\cos\psi)\,\mathbf{e}_x + (-\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi)\,\mathbf{e}_y + \sin\theta\cos\psi\,\mathbf{e}_z \\
\hat{\mathbf{e}}_3 &= \sin\theta\sin\phi\,\mathbf{e}_x - \sin\theta\cos\phi\,\mathbf{e}_y + \cos\theta\,\mathbf{e}_z\,. \qquad (4.5.25)
\end{aligned}$$

In terms of these bases the expressions for $\boldsymbol{\Omega}$ are (4.4.16) and (4.4.18), which read

$$\begin{aligned}
\boldsymbol{\Omega} &= (\dot{\theta}\,\cos\psi + \dot{\phi}\,\sin\theta\sin\psi)\hat{\mathbf{e}}_1 + (\dot{\phi}\,\sin\theta\cos\psi - \dot{\theta}\,\sin\psi)\,\hat{\mathbf{e}}_2 + (\dot{\psi} + \dot{\phi}\,\cos\theta)\,\hat{\mathbf{e}}_3 \qquad (4.5.26a) \\
&= (\dot{\theta}\,\cos\phi + \dot{\psi}\,\sin\theta\sin\phi)\mathbf{e}_x + (-\dot{\psi}\,\sin\theta\cos\phi + \dot{\theta}\,\sin\phi)\,\mathbf{e}_y + (\dot{\phi} + \dot{\psi}\,\cos\theta)\,\mathbf{e}_z\,. \qquad (4.5.26b)
\end{aligned}$$

We choose $\mathbf{e}_z$ to point in the vertical direction in the inertial frame and so these expressions imply that $\theta(t)$, which gives the angle between $\hat{\mathbf{e}}_3$ and $\mathbf{e}_z$, is also the angle between the symmetry axis and the vertical.

The kinetic energy for a symmetric top in these coordinates is given in §4.4.4, and is

$$K = \tfrac{1}{2}\,\mathcal{I}_1(\Omega_1^2 + \Omega_2^2) + \tfrac{1}{2}\,\mathcal{I}_3\Omega_3^2 = \tfrac{1}{2}\,\mathcal{I}_1(\dot{\theta}^2 + \dot{\phi}^2\,\sin^2\theta) + \tfrac{1}{2}\,\mathcal{I}_3(\dot{\psi} + \dot{\phi}\,\cos\theta)^2\,. \qquad (4.5.27)$$

Since θ is the angle between the symmetry axis and the vertical the gravitational potential energy is given by $V = Mg\ell\,\cos\theta$, leading to the Lagrangian

$$L = \tfrac{1}{2}\Big[\mathcal{I}_1(\dot{\theta}^2 + \dot{\phi}^2\,\sin^2\theta) + \mathcal{I}_3(\dot{\psi} + \dot{\phi}\,\cos\theta)^2\Big] - Mg\ell\,\cos\theta\,. \qquad (4.5.28)$$

Formal integration

Because the potential depends only on θ the Lagrangian depends on ϕ and ψ only through their time derivatives (making them ignorable coordinates). The means two immediate integrals of the equations of motion are conservation of $p_\psi = \partial L/\partial\dot\psi$ and $p_\phi = \partial L/\partial\dot\phi$ (compare with (4.4.26) and (4.4.27)). Conservation of p_ψ corresponds to the symmetry of rotations about the top's symmetry axis, $\hat{\mathbf{e}}_3$, and so states that the component of angular momentum in this direction – and so also Ω_3 – must be constant:

$$p_\psi = \frac{\partial L}{\partial\dot\psi} = \mathcal{I}_3(\dot\psi + \dot\phi\cos\theta) = \mathcal{I}_3\Omega_3 = \mathcal{I}_1 a\,, \tag{4.5.29}$$

where a is an integration constant (and the factor of $\mathcal{I}_1$ on the right-hand side is a conventional choice made for later convenience).

Conservation of p_ϕ similarly corresponds to the symmetry of rotations about the vertical, $\mathbf{e}_z$, and so states that the component of angular momentum in this direction must also be constant:

$$p_\phi = \frac{\partial L}{\partial\dot\phi} = (\mathcal{I}_1\,\sin^2\theta + \mathcal{I}_3\,\cos^2\theta)\dot\phi + \mathcal{I}_3\,\dot\psi\,\cos\theta = \mathcal{I}_1 b\,, \tag{4.5.30}$$

where b is again an integration constant and the factor of $\mathcal{I}_1$ is again conventional.

The utility of eqs. (4.5.29) and (4.5.30) is that they allow $\phi(t)$ and $\psi(t)$ to be completely solved for once $\theta(t)$ is known. Explicitly, (4.5.29) can be solved for $\dot\psi$, leading to

$$\mathcal{I}_3\dot\psi = \mathcal{I}_1 a - \mathcal{I}_3\dot\phi\cos\theta\,. \tag{4.5.31}$$

Using this to eliminate $\dot\psi$ from (4.5.30) then implies

$$\mathcal{I}_1\dot\phi\sin^2\theta + \mathcal{I}_1 a\cos\theta = \mathcal{I}_1 b \quad \text{and so} \quad \dot\phi = \frac{b - a\cos\theta}{\sin^2\theta}\,. \tag{4.5.32}$$

Using this in (4.5.31) then gives

$$\dot\psi = \frac{\mathcal{I}_1 a}{\mathcal{I}_3} - \left(\frac{b - a\cos\theta}{\sin^2\theta}\right)\cos\theta\,. \tag{4.5.33}$$

It remains to find θ and this is where another immediate integral of the motion comes in: energy conservation. The time-independent of L implies this system is invariant under time translations and so implies

$$E = K + V = \tfrac{1}{2}\,\mathcal{I}_1(\dot\theta^2 + \dot\phi^2\,\sin^2\theta) + \tfrac{1}{2}\,\mathcal{I}_3(\dot\psi + \dot\phi\,\cos\theta)^2 + Mg\ell\,\cos\theta \tag{4.5.34}$$

is also time-independent. This can be made into an expression involving only θ once $\dot\phi$ and $\dot\psi$ are eliminated using (4.5.32) and (4.5.33). In particular doing so implies the term $\frac{1}{2}\,\mathcal{I}_3(\dot\psi + \dot\phi\,\cos\theta)^2$ term is just a constant – indeed (4.5.29) shows it equals $\frac{1}{2}\,\mathcal{I}_3\Omega_3^2$. So for the purposes of solving for θ it is convenient to combine this constant with E by writing $\mathcal{E} = E - \frac{1}{2}\mathcal{I}_3\Omega_3^2$, allowing (4.5.34) to be rewritten

$$\mathcal{E} = \tfrac{1}{2}\mathcal{I}_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + Mg\ell\cos\theta\,. \tag{4.5.35}$$

Once $\dot{\phi}$ is eliminated using (4.5.32) the energy expression implies the following condition must be satisfied by $\theta(t)$:

$$\dot{\theta}^2 \sin^2\theta = (\alpha - \beta\cos\theta)\sin^2\theta - (b - a\cos\theta)^2\,, \tag{4.5.36}$$

where α and β are the following combinations of parameters:

$$\alpha := \frac{2\mathcal{E}}{\mathcal{I}_1} \quad \text{and} \quad \beta := \frac{2Mg\ell}{\mathcal{I}_1} > 0\,. \tag{4.5.37}$$

Eq. (4.5.36) can be integrated to give $\theta(t)$ and when doing so it is useful to change variables to $u(t) = \cos\theta(t)$ so (4.5.36) becomes

$$\dot{u}^2 = (1 - u^2)(\alpha - \beta u) - (b - au)^2\,. \tag{4.5.38}$$

The implicit solution for $\theta(t)$ is then given by

$$t - t_0 = \int_{u_0}^{u(t)} \frac{\mathrm{d}u}{\dot{u}} = \int_{u_0}^{u(t)} \frac{\mathrm{d}u}{\sqrt{(1-u^2)(\alpha - \beta u) - (b - au)^2}}\,. \tag{4.5.39}$$

The integral can be performed in terms of elliptic integrals and once $\theta(t)$ is obtained in this way expressions for $\phi(t)$ and $\psi(t)$ can be found by integrating eqs. (4.5.32) and (4.5.33).

Qualitative behaviour

More useful than staring at the resulting elliptic functions is acquiring a qualitative understanding of the kinds of solutions that the above equations imply. This can be most easily done starting with (4.5.38), which we write as $\dot{u}^2 = f(u)$ with $f(u)$ being the cubic polynomial in u seen on the right-hand side of (4.5.38). Because $\dot{u}^2 \geq 0$ this equation only has solutions when $f(u) \geq 0$. Furthermore since $u = \cos\theta$ we are only interested in solutions to (4.5.38) that are real and lie in the interval $-1 \leq u \leq 1$. A plot of the function $f(u)$ for a representative choice of parameters is given in Fig. (22).

For large u the function $f(u)$ is dominated by the cubic term βu^3 and because the combination of parameters appearing in β are positive – see eq. (4.5.37) – $f(u)$ is large and positive when $u \gg +1$ and large and negative when $u \ll -1$. Since $f(u)$ is cubic and changes sign in between there must therefore be one or three real roots. It is also true that when $u = \pm 1$ that $f(\pm 1) = -(b \mp a)^2 \leq 0$ and so unless $b = \pm a$ this means $u = \pm 1$ is not part of

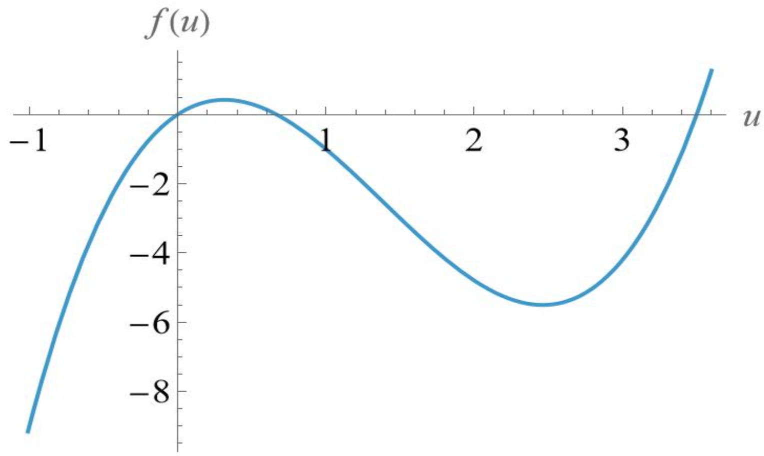

Figure 22. Plot of the function $f(u) = (1-u^2)(\alpha - \beta u) - (b - au)^2$ (for representative choices for α, β, a and b) appearing on the right-hand side of (4.5.38).

the allowed range[15] of u. But $f(u)$ is large and positive for larger u and so having $f(+1) < 0$ implies one of the roots of $f(u) = 0$ occurs for $u > +1$.

If there is any solution at all it must be that $f(u) = 0$ has three roots and two of them, $u_\pm$, must satisfy $-1 < u_- < u_+ < 1$. The evolution of θ oscillates back and forth between the two angles $\theta_\pm$ that satisfy $u_\pm = \cos\theta_\pm$, with $\dot\theta = 0$ at the turning points. Whenever $\dot\phi \neq 0$ the top precesses about the vertical axis, since changes to ϕ describe rotations about the vertical axis. Unlike for the free symmetric top, the symmetry axis *nutates* by bobbing up and down as it precesses in this way.

The qualitative behaviour of the motion divides up into three categories, depending on whether or not $\dot\phi$ ever changes sign. Eq. (4.5.32) shows that $\dot\phi$ is positive when $u < u_\star$ and is negative when $u > u_\star$ where

$$u_\star := \frac{b}{a}\,. \tag{4.5.40}$$

So there are two categories of motion depending on whether or not $u_\star$ lies between u_- and u_+. The categories can be illustrated by plotting (θ, ϕ) on the surface of a sphere, since these two coordinates provide the spherical polar angles of the direction along the top's symmetry axis (as is done in Fig. 23). If $u_\star$ does not lie in this range then $\dot\phi$ never changes sign and so the top's precession always proceeds in the same direction as it nutates up and down. This option is illustrated in panel (a) of Fig. 23. If, on the other hand, $u_- < u_\star < u_+$ then the precession is in one direction for one range of θ and changes sign for the rest of the range.

[15]The case $b = \pm a$ corresponds to when the top points vertically up or vertically down, as can be seen because eqs. (4.5.29) and (4.5.30) show that $\mathcal{I}_1 a$ is the component of angular momentum along the top's symmetry axis and $\mathcal{I}_1 b$ is the angular momentum along the vertical and these coincide (or have opposite sign) when the top points vertically up (or down).

This option is illustrated in panel (b) of Fig. 23. At the borderline case $\dot{\phi}$ goes to zero just at the edge of θ's allowed range. This option is illustrated in panel (c) of Fig. 23.

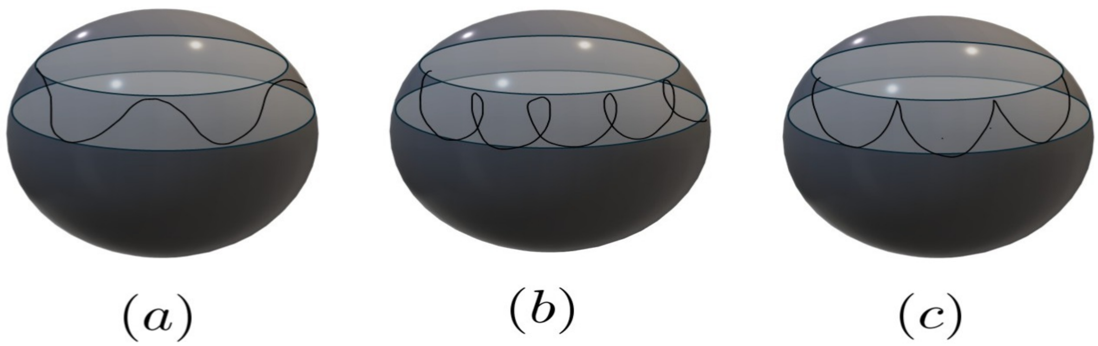

Figure 23. Trace of the direction of the symmetry direction of the nutating symmetric top with the three qualitative situations illustrated in the three panels. In these figures $\mathbf{e}_z$ defines the direction of the North Pole and the spherical polar angles (θ, ϕ) indicate the direction towards which the top's symmetry direction points. Panel (a) illustrates the case where ϕ increases monotonically and so the direction of precession does not change even as it nutates. Panel (b) illustrates the case where the direction of precession back-tracks for some ranges of the polar angle θ. Panel (c) describes the borderline case where $\dot{\phi}$ vanishes at the extreme edge of θ's nutation range.

This last case – *i.e.* case (c) – is the one that arises if the top is released from rest. With this initial condition the top initially falls downward and only then starts to precess and nutate, in an ideal world eventually returning to $\dot{\phi} = \dot{\theta} = 0$ and repeating the pattern. In practice friction at the top's point of support can damp out the nutation leading (sometimes quickly) to a straight precession in which ϕ advances for fixed θ.

Worked example: Rapidly spinning symmetric top in a magnetic field

Consider next the precession of a symmetric top consisting of charged particles with a fixed charge-to-mass ratio: $\gamma = q_a/m_a$ sitting in a constant magnetic field $\mathbf{B}$.

The kinetic energy for a symmetric top in these coordinates is given by §4.4.4, repeated again here

$$K = \tfrac{1}{2}\,\mathcal{I}_1(\Omega_1^2 + \Omega_2^2) + \tfrac{1}{2}\,\mathcal{I}_3\Omega_3^2 = \tfrac{1}{2}\,\mathcal{I}_1(\dot{\theta}^2 + \dot{\phi}^2\,\sin^2\theta) + \tfrac{1}{2}\,\mathcal{I}_3(\dot{\psi} + \dot{\phi}\,\cos\theta)^2\,. \tag{4.5.41}$$

The potential energy part of the Lagrangian describing the interaction between the top and the magnetic field is given by (2.6.22), reproduced for convenience here

$$V = \sum_a q_a\Big(\Phi - \dot{\mathbf{r}}_a \cdot \mathbf{A}\Big)\,, \tag{4.5.42}$$

where for a constant magnetic field we can take $\Phi = 0$ and $\mathbf{A} = \frac{1}{2}\,\mathbf{B} \times \mathbf{r}$ and so

$$V = -\sum_a q_a\dot{\mathbf{r}}_a \cdot \mathbf{A} = -\tfrac{1}{2}\sum_a q_a\dot{\mathbf{r}}_a \cdot (\mathbf{B} \times \mathbf{r}_a) = -\tfrac{1}{2}\sum_a q_a(\mathbf{r}_a \times \dot{\mathbf{r}}_a) \cdot \mathbf{B} = -\tfrac{1}{2}\mathbf{m} \cdot \mathbf{B}\,, \tag{4.5.43}$$

where $\mathbf{m}$ is defined by (4.5.12), and having a constant charge-to-mass ratio implies $\mathbf{m} = (Q/M)\mathbf{M}$ where $\mathbf{M}$ is the angular momentum associated with rotations about the centre of mass – see eq. (4.1.37).

We choose the top to be initially spinning along its principle axis $\hat{\mathbf{e}}_3$ so that in the absence of the magnetic field there would be no precession. We also choose $\mathbf{e}_z$ to point in the direction of the magnetic field in the inertial frame and this means that the coordinate $\theta(t)$ gives the angle between $\mathbf{M}$ and $\mathbf{B}$. The complete Lagrangian then is $L = K - V$ and so

$$L = \tfrac{1}{2}\Big[\mathcal{I}_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \mathcal{I}_3(\dot{\psi} + \dot{\phi}\cos\theta)^2\Big] + \tfrac{1}{2}\mathcal{M}B\cos\theta\,, \tag{4.5.44}$$

where $\mathcal{M} := |\mathbf{M}|$.

Following the steps for the precessing top the equations of motion for ψ and ϕ imply conservation of $p_\psi = \partial L/\partial\dot{\psi}$ and $p_\phi = \partial L/\partial\dot{\phi}$ – compare to eqs. (4.5.29) and (4.5.30):

$$p_\psi = \frac{\partial L}{\partial\dot{\psi}} = \mathcal{I}_3(\dot{\psi} + \dot{\phi}\cos\theta) = \mathcal{I}_3\Omega_3 = \mathcal{I}_1 a\,, \tag{4.5.45}$$

and

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = (\mathcal{I}_1\sin^2\theta + \mathcal{I}_3\cos^2\theta)\dot{\phi} + \mathcal{I}_3\,\dot{\psi}\cos\theta = \mathcal{I}_1 b\,, \tag{4.5.46}$$

where a and b are integration constants. Solving these for $\dot{\psi}$ and $\dot{\phi}$ then gives

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \quad \text{and} \quad \dot{\psi} = \frac{\mathcal{I}_1 a}{\mathcal{I}_3} - \left(\frac{b - a\cos\theta}{\sin^2\theta}\right)\cos\theta\,. \tag{4.5.47}$$

The energy $\mathcal{E} = E - \frac{1}{2}\mathcal{I}_3\Omega_3^2$ then is

$$\mathcal{E} = \tfrac{1}{2}\mathcal{I}_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - \tfrac{1}{2}\mathcal{M}B\cos\theta\,, \tag{4.5.48}$$

Once $\dot{\phi}$ is eliminated using (4.5.32) the energy expression implies the following condition must be satisfied by $\theta(t)$:

$$\dot{\theta}^2\sin^2\theta = (\alpha + \beta\cos\theta)\sin^2\theta - (b - a\cos\theta)^2\,, \tag{4.5.49}$$

where α and β are the following combinations of parameters:

$$\alpha := \frac{2\mathcal{E}}{\mathcal{I}_1} \quad \text{and} \quad \beta := \frac{2\mathcal{M}B}{\mathcal{I}_1} > 0\,. \tag{4.5.50}$$

Notice that the sign of the β term in (7.1.45) differs from what was found in (4.5.36) in the case of a gravitational torque.

The analysis now proceeds as in the gravitational case by solving (4.5.50) for $\theta(t)$. The qualitative behaviour is easier to see if we change variables (as before) to $u = \cos\theta$, so (4.5.50) becomes

$$\dot{u}^2 = (1 - u^2)(\alpha + \beta u) - (b - au)^2 =: \tilde{f}(u)\,, \tag{4.5.51}$$

where the last equality defines $\tilde{f}(u)$. Because $\dot{u}^2 \geq 0$ this equation only has solutions when $\tilde{f}(u) \geq 0$, and we only care about the solutions that lie in the interval $-1 \leq u \leq 1$. Although the change in sign of the β term changes the asymptotic form of $\tilde{f}(u)$ for large positive and negative u this does not qualitatively change the discussion of the range of angles ϕ and θ are allowed to pass through.

The resulting motion is similar to what was found for the gravitational case, inasmuch as it in general describes a combination of precession and nutation (with nutation being motion where θ changes with time), and the three categories illustrated in Fig. 23 also apply here.

⋆ ⋆ ⋆

5 Harmonic Motion

Simple harmonic motion – defined in its simplest form in §1.1.1 – is perhaps the most studied physics problem there is. This is not just because it can be solved so explicitly; it is also familiar because it appears in a great many physical systems. There is a very good reason for this: harmonic motion arises whenever a restoring force is linear in the displacement from an equilibrium position, but even very complicated position-dependent forces are well-approximated by a linear dependence on position if restricted to very small displacements.

This makes simple harmonic motion a natural next step in our discussion of macroscopic bodies. We saw in §1.3.1 that the centre of mass position provides the most coarse-grained description of a macroscopic object built from many smaller constituent atoms. §4 then argued that the next most coarse-grained is the rigid-body motion that describes how objects behave in the limit that internal motion of the constituent atoms is negligible so the distance between their atoms is fixed.

The next step as we zoom in to describe macroscopic objects with successively more detail allows the internal atoms to deviate by a small distance away from their equilibrium position, and as we see in this section for small enough deviations this motion is well-approximated by simple harmonic motion. Along the way we see in a more quantitative way in what sense rigid-body motion and centre-of-mass motion emerge as being the most important at low energies.

5.1 Two-body oscillations

Before describing systems with N constituent particles moving in three dimensions we start with the simplest case where $N = 2$. To this end we specialize here to the motion of two atoms, with masses m_1 and m_2, bound together into a diatomic molecule.

We furthermore assume the two atoms interact through rotationally invariant conservative forces and that these forces predict the atoms minimize their energy when they are separated by a fixed distance: $r = |\mathbf{r}_1 - \mathbf{r}_2| = \ell$ (see Fig. 24). In real molecules this happens because the forces between neutral atoms are typically weakly attractive for intermediate separations but become strongly repulsive for small separations.

In detail the weak attraction occurs because atoms are built from electrically charged constituents whose electrostatic fields do not perfectly cancel because the constituents are not at exactly the same positions. The mobility of these constituents allows them to adjust their relative positions to maximize the attraction between opposite charges situated on different atoms. The strong repulsion arises because the most mobile constituents – the electrons – are fermions whose quantum statistics forbids them from sitting on top of one another. A sketch of the resulting potential energy as a function of interatomic separation is given in Fig. 25.

Given the potential $V(r)$ Newton's 2nd law for the two atoms becomes

$$m_1\ddot{\mathbf{r}}_1 = -\nabla_1 V = -V'(r)\,\mathbf{e}_r \quad \text{and} \quad m_2\ddot{\mathbf{r}}_2 = -\nabla_2 V = V'(r)\,\mathbf{e}_r\,, \tag{5.1.1}$$

Figure 24. A simple diatomic molecule consisting of two atoms (with mass m_1 and m_2 respectively) with equilibrium separation ℓ.

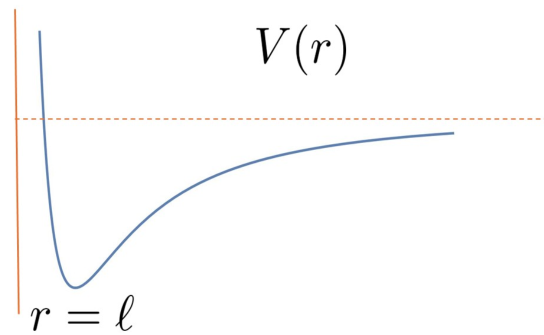

Figure 25. A sketch of the potential energy $V(r)$ as a function of separation whose minimization determines the separation ℓ seen in Fig. 24.

where $\mathbf{e}_r = \mathbf{r}/r$ is the unit vector pointing in the direction $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. The centre of mass motion is decoupled by switching from $\mathbf{r}_1$ and $\mathbf{r}_2$ to $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/M$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, where $M = m_1 + m_2$, because – as we saw in §1.2.1 – eqs. (5.1.1) become

$$M\ddot{\mathbf{R}} = 0 \quad \text{and} \quad \mu\,\ddot{\mathbf{r}} = -V'(r)\,, \tag{5.1.2}$$

where $\mu = m_1 m_2/M$.

Restriction to small oscillations

Now comes one of the main points: if we restrict ourselves to situations where $r = |\mathbf{r}|$ is very close to ℓ then we can Taylor expand the potential in powers of $\mathbf{r} - \mathbf{r}_0$ (where $\mathbf{r}_0$ has length ℓ). This expansion starts at second order because $\mathbf{r}_0$ is a stationary point: $V'(r = \ell) = 0$. In this case the first few terms of the expansion are

$$V(r) = V(\ell) + (x - x_0)^i \Big(\partial_i V\Big)_{\mathbf{r}=\mathbf{r}_0} + \tfrac{1}{2}\,(x - x_0)^i\,(x - x_0)^j \Big(\partial_i\partial_j V\Big)_{\mathbf{r}=\mathbf{r}_0} + \cdots\,, \tag{5.1.3}$$

where we write the components as $\mathbf{r} = x^i\,\mathbf{e}_i$ and $\mathbf{r}_0 = x_0^i\,\mathbf{e}_i$ and $\partial_i V = \partial V/\partial x^i$. For any arbitrary function of r the derivatives evaluate to

$$\partial_i V = V'(r)\,\frac{x_i}{r} \quad \text{and} \quad \partial_j \partial_i V = V''(r)\,\frac{x_i x_j}{r^2} + V'(r)\left[\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}\right], \tag{5.1.4}$$

and so on, so evaluating at $\mathbf{r} = \mathbf{r}_0$ where $|\mathbf{r}_0| = \ell$ and $V'(\ell) = 0$ implies

$$(\partial_i V)_{\mathbf{r}=\mathbf{r}_0} = 0 \quad \text{and} \quad (\partial_i \partial_j V)_{\mathbf{r}=\mathbf{r}_0} = V''(\ell)\frac{x_{0i}x_{0j}}{\ell^2}\,. \tag{5.1.5}$$

Because $r = \ell$ is a local minimum of $V(r)$ – see Fig. 25 – we know $V''(\ell) > 0$.

Combining everything allows (5.1.3) to be written

$$V(r) = V(\ell) + \frac{V''(\ell)}{2\ell^2}\Big[\mathbf{r}_0 \cdot (\mathbf{r} - \mathbf{r}_0)\Big]^2 + \cdots\,. \tag{5.1.6}$$

Keeping only the quadratic term in V implies the force appearing in the equation of motion (5.1.2) for $\mathbf{r}$ is linear in $\mathbf{y} := \mathbf{r} - \mathbf{r}_0$, taking the form

$$\mu\ddot{\mathbf{y}} = -\frac{V''(\ell)}{\ell^2}(\mathbf{y}\cdot\mathbf{r}_0)\,\mathbf{r}_0\,. \tag{5.1.7}$$

The complicated matrix form on the right-hand side can be simplified by decomposing $\mathbf{y}$ into a piece parallel to $\mathbf{r}_0$ and a piece perpendicular to $\mathbf{r}_0$. Choosing a basis of orthonormal unit vectors $\mathbf{e}_i$ where $\mathbf{e}_3 = \mathbf{r}_0/\ell$ and $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ we can write

$$\mathbf{y}(t) = y_1(t)\,\mathbf{e}_1 + y_2(t)\,\mathbf{e}_2 + y_3(t)\,\mathbf{e}_3\,, \tag{5.1.8}$$

and so (5.1.7) becomes three uncoupled equations:

$$\mu\ddot{y}_1 = \mu\ddot{y}_2 = 0 \quad \text{and} \quad \mu\ddot{y}_3 = -V''(\ell)\,y_3\,. \tag{5.1.9}$$

All three of these have the simple harmonic oscillator form, $\ddot{y} + \omega^2 y = 0$ where

$$\omega = 0 \quad \text{for } y_1 \text{ and } y_2 \quad \text{and} \quad \omega^2 = \frac{V''(\ell)}{\mu} \quad \text{for } y_3. \tag{5.1.10}$$

The variables y_1 and y_2 with $\omega = 0$ are called *zero modes* and because they have no potential barrier and zero frequency they are not really harmonic oscillators. These degrees of freedom we've seen before: they are the rotational degrees of freedom of the diatomic molecule for rotations about two axes perpendicular to the line connecting the two atoms. This can be seen because a rotation $\delta\mathbf{r} = \boldsymbol{\Omega} \times \mathbf{r}_0$ for some $\boldsymbol{\Omega}$ is always perpendicular to $\mathbf{r}_0$. There is no potential barrier for these rotation directions because rotations are (by assumption) a symmetry of the interatomic interactions and so they do not change the potential energy $V(r)$.

Because they are rotations their motion can be understood without having to Taylor expand the potential energy about $\mathbf{r} = \mathbf{r}_0$. Their dynamics is precisely the dynamics of rigid body motion and so – as we've seen in §4 – their equations of motion are already encoded into the conservation law for angular momentum, $\dot{\mathbf{J}} = 0$, where in this instance

$$\mathbf{J} = m_1\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2\mathbf{r}_2 \times \dot{\mathbf{r}}_2 = M\mathbf{R} \times \dot{\mathbf{R}} + \mathbf{M}\,, \tag{5.1.11}$$

where

$$\mathbf{M} = m_1\mathbf{s}_1 \times \dot{\mathbf{s}}_1 + m_2\mathbf{s}_2 \times \dot{\mathbf{s}}_2 \tag{5.1.12}$$

and $\mathbf{s}_i := \mathbf{r}_i - \mathbf{R}$ is the displacement of each atom from the molecule's centre of mass. Only two components of $\mathbf{M}$ contain nontrivial evolution information because the molecule is a rotor, in the sense defined in (4.1.22): because the atoms are collinear there is no meaning to rotations about the line connecting them.

For these reasons we need not pursue further the zero modes in (5.1.9) and (5.1.10) and instead focus purely on the dynamics of y_3, which satisfies

$$\ddot{y}_3 + \omega^2\, y_3 = 0 \quad \text{with} \quad \omega^2 = \frac{V''(\ell)}{\mu}\,. \tag{5.1.13}$$

The solution is

$$y_3(t) = A\,\cos(\omega t + \delta)\,, \tag{5.1.14}$$

where the integration constants A and δ are determined by the initial conditions $y_3(0)$ and $\dot{y}_3(0)$. For very small deviations from equilibrium the relative motion of the two atoms along the direction connecting them is described by simple harmonic motion, while motion relative to the centre of mass in the two directions perpendicular to the molecule's axis are described by the rigid motion of a rotator (for which $\mathcal{I}_3 = 0$).

From this point of view we can see more precisely how the rigid body approximation arises: it is obtained in the limit that the potential becomes extremely steeply sloped about its minimum so that small oscillations cost a lot of energy. This is the limit $V''(\ell) \to \infty$ or equivalently $\omega \to \infty$, in which case oscillations are all much too rapid to be relevant to the low-energy world. *Any* object whose atoms have equilibrium positions relative to one another and whose motion is studied only over timescales much longer than characteristic oscillation frequencies (like ω in this example) can be regarded as rigid bodies. They can be regarded as rigid bodies combined with harmonic motion if the atomic displacements from equilibrium are sufficiently small.

5.2 Coupled Oscillators

The case of N atoms with small displacements from equilibrium shares many of the features of the $N = 2$ special case. In particular, if the interatomic forces are rotation invariant then there are always three 'zero modes' corresponding to rotations whose dynamics captures rigid body motion. There are three zero modes rather than two in the general case because there are in general three independent rotations (although only two of these matter for the special case when all atoms lie along a line since rotations about the line do not change the atomic configuration).

Once these and the centre of mass motion are excluded the remaining $3N-6$ degrees of freedom are well-described by a system of coupled harmonic oscillators for motion sufficiently close to the equilibrium positions. To see how this emerges explicitly we extend the derivation made above for diatomic molecules to the case of general N. To start with let us suppose that no external force is applied to the object, $\mathbf{F}_a^{\rm ext}=0$. Then eq. (1.3.3) ensures the centre of mass does not accelerate: $\ddot{\mathbf{R}}=0$ and we can choose an inertial frame for which $\mathbf{R}=0$, and so for which there is no distinction between an atom's inertial-frame position $\mathbf{r}_a$ and its displacement $\mathbf{s}_a=\mathbf{r}_a-\mathbf{R}$ relative to the centre of mass. Newton's 2nd law is then given by (1.3.1) specialized to the case where $\mathbf{F}_a^{\rm ext}=0$:

$$\begin{aligned} m_1\,\ddot{\mathbf{s}}_1 &= \qquad\; + \mathbf{F}_{12}+\mathbf{F}_{13}+\cdots+\mathbf{F}_{1N} = \mathbf{F}_1^{\rm tot} \\ m_2\,\ddot{\mathbf{s}}_2 &= \mathbf{F}_{21} \qquad\; + \mathbf{F}_{23}+\cdots+\mathbf{F}_{2N} = \mathbf{F}_2^{\rm tot} \\ m_3\,\ddot{\mathbf{s}}_3 &= \mathbf{F}_{31}+\mathbf{F}_{32}+ \qquad \cdots+\mathbf{F}_{3N} = \mathbf{F}_3^{\rm tot} \\ \vdots\;\; &= \\ m_N\,\ddot{\mathbf{s}}_N &= \mathbf{F}_{N1}+\mathbf{F}_{N2}+\mathbf{F}_{N3}+\cdots \qquad = \mathbf{F}_N^{\rm tot}\,, \end{aligned} \tag{5.2.1}$$

where the last equality on each line defines the net force

$$\mathbf{F}_a^{\rm tot}(\mathbf{s}_1,\cdots,\mathbf{s}_N) := \sum_{b\neq a}\mathbf{F}_{ab}\,, \tag{5.2.2}$$

applied to each atom, regarded as a function of the atomic positions.

The key assumption is that there exists an equilibrium configuration, $\overline{\mathbf{s}}_a$, for the $\mathbf{s}_a$'s, in the sense that if the atoms are started off at rest with $\mathbf{s}_a=\overline{\mathbf{s}}_a$ then the net force on each atom vanishes when evaluated at the equilibrium configuration:

$$\mathbf{F}_a^{\rm tot}(\overline{\mathbf{s}}_1,\cdots,\overline{\mathbf{s}}_N)=0 \quad \hbox{for all } a\,. \tag{5.2.3}$$

This ensures that (5.2.1) implies $\ddot{\mathbf{s}}_a=0$ at this configuration (as is required if it is to be a static solution to the equations of motion).

If all atoms are displaced from this equilibrium by a nonzero amount $\mathbf{y}_a=\mathbf{s}_a-\overline{\mathbf{s}}_a=\mathbf{r}_a-\overline{\mathbf{r}}_a$ then these forces no longer exactly cancel. Their nonzero sum can be Taylor expanded in powers of the components of $\mathbf{y}_a$ and this expansion starts at linear order[16] in $\mathbf{y}_a$ because the net force vanishes when all of the $\mathbf{y}_a$'s vanish. That is

$$\mathbf{F}_a^{\rm tot}(\overline{\mathbf{s}}_1+\mathbf{y}_1,\cdots,\overline{\mathbf{s}}_N+\mathbf{y}_N)\simeq\sum_{b=1}^N k_{ab}\,\mathbf{y}_b\,, \tag{5.2.4}$$

[16]Strictly speaking, having the expansion starts at linear order assumes the total force is analytic in the separations at $\mathbf{y}_a=0$.

where for each a and b the coefficient k_{ab} is a 3×3 matrix corresponding to the 3 components of the vectors $\mathbf{y}_b$ and $\mathbf{F}_a^{\text{tot}}$. Notice that Newton's third law $\mathbf{F}_{ab} = -\mathbf{F}_{ba}$ (and the assumed absence of a net external force) implies

$$\sum_{a=1}^{N} \mathbf{F}_a^{\text{tot}} = 0 \quad \text{and so} \quad \sum_{a=1}^{N} \sum_{b=1}^{N} k_{ab}\, \mathbf{y}_b = 0 \tag{5.2.5}$$

which is a restriction on the solutions $\mathbf{y}_a$.

Following the practice of §2.2 it is useful to combine the indices $a = 1, \cdots, N$ and $i = x, y, z$ into the single index $A = \{a, i\}$ or $B = \{b, j\}$ that runs from 1 to $3N$. In terms of this the matrix of coefficients is given by $k_{AB} = \Big(k_{ab}\Big)_{ij} = \partial F_{ai}^{\text{tot}} / \partial y_{bj} = \partial F_A^{\text{tot}} / \partial y_B$, and so

$$k_{AB} = \frac{\partial F_{ai}^{\text{tot}}}{\partial y_{bj}} = \sum_{c \neq a} \frac{\partial (\mathbf{F}_{ac})_i}{\partial y_{bj}} \,. \tag{5.2.6}$$

In the event that the interatomic forces are conservative they can be obtained by differentiating a potential energy:

$$\mathbf{F}_a^{\text{tot}} = -\nabla_a U \quad \text{where} \quad U = U(\mathbf{s}_1, \cdots, \mathbf{s}_N) \,. \tag{5.2.7}$$

Using this in (5.2.6) shows that for conservative forces the matrix k_{AB} is real and symmetric

$$k_{AB} = -\frac{\partial^2 U}{\partial y_{ai} \partial y_{bj}} = -\frac{\partial^2 U}{\partial y^A \partial y^B} = k_{BA} \,. \tag{5.2.8}$$

In the conservative case the assumption that $\mathbf{F}_a^{\text{tot}}$ vanishes for a configuration $\bar{\mathbf{s}}_a$ and is well approximated by terms linear in the deviation $\mathbf{y}_a = \mathbf{s}_a - \bar{\mathbf{s}}_a$ – as in (5.2.4) – is equivalent to the assumption that $U(\mathbf{s}_1, \cdots, \mathbf{s}_N)$ has a local minimum for $\mathbf{s}_a = \bar{\mathbf{s}}_a$ and then approximating U by its leading term in the Taylor expansion about this minimum. Because the expansion is performed about a minimum we know $\partial U / \partial y^A = 0$ and so the leading term is quadratic in the fluctuations.

It is tempting to argue that because the expansion is about a minimum (as opposed to a maximum, say) then the matrix k_{AB} must also only have strictly positive eigenvalues, though this is not actually true. If the interatomic forces are rotationally invariant (and if any of the equilibrium positions $\bar{\mathbf{s}}_a$ are nonzero) then we know that there must be a zero eigenvalue corresponding to each of the directions obtained by rotating $\{\bar{\mathbf{s}}_a\} \to \{R\, \bar{\mathbf{s}}_a\}$ for some 3×3 rotation matrix R.

These must give eigenvectors of k_{AB} with zero eigenvalue for the same reason as was found above in §5.1: the rotation invariance of the potential U. To see why, notice that rotation invariance of the potential implies

$$U(\mathbf{s}_1 + \delta\mathbf{s}_1, \cdots, \mathbf{s}_N + \delta\mathbf{s}_N) \equiv U(\mathbf{s}_1, \cdots, \mathbf{s}_N) \tag{5.2.9}$$

for arbitrary $\mathbf{s}_a$ and for each of the three independent infinitesimal rotations $\delta\mathbf{s}_a$ of $\mathbf{s}_a$. Taylor expanding this in powers of $\delta\mathbf{s}_a$ implies at linear order

$$\frac{\partial U}{\partial s^A}\,\delta s^A = 0 \qquad \text{(implied sum on A)}\,, \tag{5.2.10}$$

holds as an identity for all $\mathbf{s}_a$ and for arbitrary rotations $\delta\mathbf{s}_a$. Differentiating this expression with respect to the components of the initial positions then implies

$$\frac{\partial^2 U}{\partial s^B \partial s^A}\,\delta s^A = 0 \qquad \text{(implied sum on A)}\,. \tag{5.2.11}$$

Now comes the main point: *if* $\delta s^A \neq 0$ then (5.2.11) implies that the vector with components δs^A is a zero eigenvector of the matrix $\partial^2 U/\partial s^A \partial s^B$. For brevity zero eigenvectors for the interaction matrix are collectively known as *zero modes* in what follows.

Evaluating this result at $\mathbf{s}_a = \bar{\mathbf{s}}_a$ then shows that if $\delta\bar{\mathbf{s}}_a \neq 0$ then this rotation is therefore a zero eigenvector of the coefficient matrix k_{AB}. For most objects there are three independent zero eigenvectors that can be built in this way corresponding to the three independent spatial rotations, though there can be exceptions where fewer zero eigenvectors arise (typically for rotors, such as the diatomic molecule) if there are rotations for which $\delta\bar{\mathbf{s}}_a = 0$ for all a.

In retrospect, a similar story also goes through for spatial translations. If we had done the analysis using the original variables $\mathbf{r}_a$ rather than removing the centre of mass position then we'd also expect to find three zero modes associated with translations of the equilibrium positions $\bar{\mathbf{r}}_a$. Since we have to deal with zero modes assocated with rotations anyway, from here on we revert to using the original coordinates $\mathbf{r}_a$ (rather than $\mathbf{s}_a$) and find the oscillatory modes by projecting onto the space of deviations $\mathbf{y}_a$ that are orthogonal to the symmetry-driven zero modes.

5.2.1 Normal Modes

Writing $\mathbf{r}_a = \bar{\mathbf{r}}_a + \mathbf{y}_a$ and using the small-$\mathbf{y}_a$ approximation for the restoring force in (5.2.1) gives the implications of Newton's laws for the evolution of the displacements $\mathbf{y}_a$:

$$m_a\,\ddot{\mathbf{y}}_a = \mathbf{F}_a^{\rm tot} \simeq \sum_{b=1}^{N} k_{ab}\,\mathbf{y}_b \quad \text{(no sum on } a)\,, \tag{5.2.12}$$

where $\ddot{\mathbf{r}}_a$ can be replaced with $\ddot{\mathbf{y}}$ if the centre of mass and the equilibrium positions, $\bar{\mathbf{r}}_a$, are time independent. The approximate equality in (5.2.12) assumes the deviations $\mathbf{y}_a$ are sufficiently small that terms involving two or more powers of $\mathbf{y}_a$ can be ignored.

Equivalently, the evolution of the components y^A satisfy

$$m_{AB}\ddot{y}^B = k_{AB}\,y^B \quad \text{(implied sum on } B)\,, \tag{5.2.13}$$

where we define the mass matrix

$$m_{AB} = m_{ai,bj} = m_a\,\delta_{ab}\,\delta_{ij} = m_{BA}\,. \tag{5.2.14}$$

Clearly this definition implies m_{AB} is also real and symmetric (and so Hermitian) with strictly positive eigenvalues $m_a > 0$.

Eqs. (5.2.13) are a set of $3N$ coupled differential equations that are to be solved to determine how the components $y^A(t)$ evolve in time, though we here project out the 6 'trivial' solutions corresponding to spatial translations and rigid rotations about the centre of mass. The remaining degrees of freedom define the problem of *coupled harmonic oscillators* and because their coupled evolution equations are linear the solution to how they move can be found very explicitly.

The solution is easiest to describe if (5.2.13) is written in matrix form:

$$\mathbb{M}\,\ddot{\mathbb{Y}} = \mathbb{K}\,\mathbb{Y}\,, \tag{5.2.15}$$

where $\mathbb{M}$ and $\mathbb{K}$ are the real symmetric matrices with elements m_{AB} and k_{AB} respectively and $\mathbb{Y}$ is the column vector with components y^A. As described above we expect $\mathbb{Y}$ to include zero modes due to translational and rotational symmetries and the *bona fide* oscillation spectrum is found by choosing the vector $\mathbb{Y}$ to be perpendicular to these zero modes. We assume the equilibrium positions occur at a local minimum of U and so all of the remaining eigenvalues are positive.[17]

Because the matrix $\mathbb{M}$ has strictly positive eigenvalues it is invertible and its square root can be defined.[18] So we can always define $\mathbb{Y} = \mathbb{M}^{-1/2}\,\widetilde{\mathbb{Y}}$ since $\mathbb{M}^{-1/2}$ always exists because the eigenvalues of $\mathbb{M}^{1/2}$ are all positive. In terms of this eq. (5.2.15) becomes

$$\mathbb{M}^{1/2}\,\ddot{\widetilde{\mathbb{Y}}} = \mathbb{K}\,\mathbb{M}^{-1/2}\widetilde{\mathbb{Y}}\,. \tag{5.2.16}$$

Multiplying this equation through on the left by $\mathbb{M}^{-1/2}$ then gives

$$\ddot{\widetilde{\mathbb{Y}}} = \mathbb{A}\,\widetilde{\mathbb{Y}} \qquad \text{where } \mathbb{A} := \mathbb{M}^{-1/2}\,\mathbb{K}\,\mathbb{M}^{-1/2}\,. \tag{5.2.17}$$

Recall now that $\mathbb{M}$ is real and diagonal with positive entries and so in particular is real and symmetric in the sense that $\mathbb{M}^T = \mathbb{M}$. The matrix $\mathbb{K}$ is also real and symmetric: $\mathbb{K}^T = \mathbb{K}$. This implies the matrix $\mathbb{A}$ must itself be real and symmetric. It is real because it is a product of real matrices and it is symmetric because

$$\mathbb{A}^T = (\mathbb{M}^{-1/2})^T\mathbb{K}^T(\mathbb{M}^{-1/2})^T = \mathbb{M}^{-1/2}\,\mathbb{K}\,\mathbb{M}^{-1/2} = \mathbb{A}. \tag{5.2.18}$$

[17] If the matrix $\mathbb{K}$ has zero eigenvectors besides those due to the invariance of U under spatial translations and rotations then the equilibrium configuration $\bar{\mathbf{s}}_a$ describe a saddle point of U rather than a local minimum and are said to be only *marginally stable*. We define a *stable equilibrium* to be one located at a local minimum of U for which no zero (or negative) eigenvalues arise (beyond the trivial zero eigenvalues that follow on symmetry grounds).

[18] In the present instance $\mathbb{M}^{1/2}$ is simply the matrix $\mathbb{M}$ where the diagonal entries m_a are everywhere replaced by $\sqrt{m_a}$.

It follows that $\mathbb{A}$ can be diagonalized by an orthogonal transformation, which means there exists an orthogonal matrix $\mathbb{O}$ for which

$$\mathbb{O}^\dagger \mathbb{A} \mathbb{O} = \mathbb{D} = \text{diag}\Big(\omega_1^2, \omega_2^2, \cdots \omega_{3N}^2\Big) \tag{5.2.19}$$

is diagonal with real and non-negative diagonal entries (3 zero entries corresponding to the rotational zero modes plus other positive entries).

The solution to (5.2.15) is then found by redefining variables so that $\widetilde{\mathbb{Y}} = \mathbb{O}\,\widehat{\mathbb{Y}}$ since then (5.2.17) becomes

$$\mathbb{O}\ddot{\widehat{\mathbb{Y}}} = \mathbb{A}\,\mathbb{O}\,\widehat{\mathbb{Y}}\,, \tag{5.2.20}$$

or, after multiplying through on the left by $\mathbb{O}^T = \mathbb{O}^{-1}$:

$$\ddot{\widehat{\mathbb{Y}}} = \mathbb{O}^T \mathbb{A}\,\mathbb{O}\,\widehat{\mathbb{Y}} = \mathbb{D}\,\widehat{\mathbb{Y}}\,. \tag{5.2.21}$$

Here $\mathbb{D}$ on the right-hand side is the diagonal matrix with non-negative entries defined in (5.2.19), so (5.2.21) has decoupled the differential equations.

Denoting the components of $\widehat{\mathbb{Y}}$ by $\hat{y}^A$, eq. (5.2.21) shows that each of the $\hat{y}^A$ variables satisfies its own ordinary differential equation:

$$\ddot{\hat{y}}^A + \omega_A^2\,\hat{y}^A = 0 \qquad (\text{no sum on } A)\,, \tag{5.2.22}$$

and so is an elementary simple harmonic oscillator. Each of the components $\hat{y}^A$ is called a *normal mode* of the coupled oscillator system. The general solution therefore is

$$\hat{y}^A(t) = \mathcal{A}^A \cos(\omega_A t + \delta_A) \qquad (\text{no sum on } A)\,. \tag{5.2.23}$$

Each normal mode oscillates independently with its own specific characteristic frequency.

The solution in terms of the original coupled variables is then found by concatenating the redefinitions for $\mathbb{Y}$ in terms of $\widetilde{\mathbb{Y}}$ and $\widehat{\mathbb{Y}}$, leading to

$$\mathbb{Y}(t) = \mathbb{M}^{-1/2}\,\widetilde{\mathbb{Y}}(t) = \mathbb{M}^{-1/2}\,\mathbb{O}\,\widehat{\mathbb{Y}}(t)\,. \tag{5.2.24}$$

The general solution is therefore a superposition of normal modes whose amplitudes are weighted by the overlap between the initial conditions and each normal mode.

To see how this works in detail it is worth working through a few simple examples.

Worked example: Diatomic Molecule revisited

Consider the diatomic molecule of length ℓ described in §5.1 and illustrated in Fig. 24. Let us choose the z axis parallel to the line connecting the two atoms in their equilibrium positions with the origin chosen to sit at the centre of mass, so

$$\bar{\mathbf{s}}_1 = \frac{m_2 \ell}{M}\,\mathbf{e}_z \quad \text{and} \quad \bar{\mathbf{s}}_2 = -\frac{m_1 \ell}{M}\,\mathbf{e}_z\,, \tag{5.2.25}$$

where (as usual) $M = m_1 + m_2$ is the sum of the two atomic masses. Notice $m_1\bar{\mathbf{s}}_1 + m_2\bar{\mathbf{s}}_2 = 0$ – as must happen in the centre of mass frame – and $\mathbf{s}_1 - \mathbf{s}_2 = \ell\,\mathbf{e}_z$.

To map onto the general process outlined above we do not project out the centre of mass motion and so the atomic displacements are $\mathbf{r}_a = \bar{\mathbf{r}}_a + \mathbf{y}_a$, where translation and rotation invariance of the potential energy implies it depends only on the interatomic distance $V = V(r)$ where $r = |\mathbf{r}_1 - \mathbf{r}_2| = |\mathbf{s}_1 - \mathbf{s}_2| = |\ell\,\mathbf{e}_z + \mathbf{y}_1 - \mathbf{y}_2|$. We assume as before that $V(r)$ has a local minimum at $r = \ell$ and so $V'(\ell) = 0$ and $V''(\ell) > 0$.

In this case there are six coordinates, y^A, with $A = \{a, i\}$ and $a = 1, 2$ and $i = x, y, z$, so

$$\mathbb{Y} = \begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \\ y^5 \\ y^6 \end{pmatrix} = \begin{pmatrix} y_{1x} \\ y_{1y} \\ y_{1z} \\ y_{2x} \\ y_{2y} \\ y_{2z} \end{pmatrix}, \tag{5.2.26}$$

and

$$r(\mathbf{y}_1, \mathbf{y}_2) = \sqrt{(y_{1x} - y_{2x})^2 + (y_{1y} - y_{2y})^2 + (y_{1z} - y_{2z} + \ell)^2} \tag{5.2.27}$$

so

$$-\frac{\partial V}{\partial y_{2i}} = \frac{\partial V}{\partial y_{1i}} = V'(r)\,\frac{\partial r}{\partial y_{2i}} = V'(r)\left(\frac{y_{1i} - y_{2i} + \ell\,\delta_{iz}}{r}\right). \tag{5.2.28}$$

Using $r(\mathbf{y}_1 = \mathbf{y}_2 = 0) = \ell$ and $V'(\ell) = 0$ shows $\partial V/\partial y_{ai}$ vanishes when $\mathbf{y}_a = 0$ (as it must when evaluated at a minimum). Differentiating again and evaluating at $\mathbf{y}_a = 0$ also shows that the only nonzero second derivatives at the minimum are

$$\left(\frac{\partial^2 V}{\partial y_{1z}^2}\right)_{y^A=0} = \left(\frac{\partial^2 V}{\partial y_{2z}^2}\right)_{y^A=0} = -\left(\frac{\partial^2 V}{\partial y_{1z}\partial y_{2z}}\right)_{y^A=0} = V''(\ell)\,, \tag{5.2.29}$$

so the matrix $k_{AB} = (\partial^2 V/\partial y^A \partial y^B)_{\mathbf{y}_a=0}$ is given by

$$\mathbb{K} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} V''(\ell)\,. \tag{5.2.30}$$

It is useful in what follows to separate out the spatial components x, y, z from the atomic labels a, b in this matrix, which we can do by writing it as

$$\mathbb{K} = \kappa \otimes \sigma \quad \text{or} \quad k_{ai,bj} = \kappa_{ab}\,\sigma_{ij} \tag{5.2.31}$$

where $\sigma_{ij} = \delta_{iz}\,\delta_{jz}$ so

$$\kappa = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} V''(\ell) \quad \text{and} \quad \sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.2.32}$$

This matrix has three zero eigenvalues corresponding to infinitesimal translations in each of the x, y and z directions: $\delta\mathbf{y}_1 = \delta\mathbf{y}_2 = \boldsymbol{\epsilon}$. The corresponding normalized zero eigenvectors of $\mathbb{K}$ are

$$\delta_{tx}\mathbb{Y} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\\0\\0\end{pmatrix}\,, \quad \delta_{ty}\mathbb{Y} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\0\\1\\0\end{pmatrix} \quad \text{and} \quad \delta_{tz}\mathbb{Y} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\0\\0\\1\end{pmatrix}\,, \tag{5.2.33}$$

as can be verified by explicitly multiplying by $\mathbb{K}$. These zero modes arise because the 2×2 matrix

$$\kappa \propto \begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} \quad \text{has a zero eigenvector} \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\,, \tag{5.2.34}$$

due to the fact that V depends only on the translation-invariant differences $\mathbf{y}_1 - \mathbf{y}_2$.

There are also two nontrivial rotations about the centre of mass (one each for an axis parallel to the x and y directions) since the equilibrium configuration for the molecule has the two atoms lying along the z axis. These arise due to the existence of the two zero eigenvectors

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\ \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \text{of the matrix} \quad \sigma \propto \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\,, \tag{5.2.35}$$

which would have not existed if σ were to have the only possible rotation invariant form: $\sigma_{ij} \propto \delta_{ij}$. Explicitly, performing an infinitesimal rotation in the x direction of the equilibrium vectors given in (5.2.25) gives $\delta y_{1y} = \theta_x\,(m_2\ell/M)$ and $\delta y_{2y} = -\theta_x\,(m_1\ell/M)$ and performing an infinitesimal rotation about the y direction similarly gives $\delta y_{1x} = \theta_y\,(m_2\ell/M)$ and $\delta y_{1x} = -\theta_y\,(m_1\ell/M)$. The corresponding zero eigenvectors of $\mathbb{K}$ therefore are

$$\delta_{rx}\mathbb{Y} = \begin{pmatrix}0\\ m_2/M\\ 0\\ 0\\ -m_1/M\\ 0\end{pmatrix} \quad \text{and} \quad \delta_{ry}\mathbb{Y} = \begin{pmatrix}m_2/M\\ 0\\ 0\\ -m_1/M\\ 0\\ 0\end{pmatrix}\,. \tag{5.2.36}$$

Unlike the translations of eq. (5.2.33) these rotations move the two atoms in opposite directions.

By taking linear combinations of these we arrive at a convenient basis of normalized symmetry-driven zero eigenvectors:

$$\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}\,,\quad \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}\,,\quad \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}\,,\quad \begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\0\\0\\1\end{pmatrix}\,. \tag{5.2.37}$$

The normalized basis vector orthogonal to these five is an eigenvector for the nonzero eigenvalue:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\0\\0\\-1\end{pmatrix} \quad \text{has nonzero eigenvalue} \quad \lambda = 2V''(\ell) > 0\,. \tag{5.2.38}$$

The mass matrix in these coordinates similarly is

$$\mathbb{M} = \begin{pmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_2 \end{pmatrix} \quad \text{and so} \quad \mathbb{M}^{-1/2} = \begin{pmatrix} m_1^{-1/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1^{-1/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & m_1^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2^{-1/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & m_2^{-1/2} \end{pmatrix}. \tag{5.2.39}$$

The matrix $\mathbb{A} = \mathbb{M}^{-1/2}\,\mathbb{K}\,\mathbb{M}^{-1/2}$ therefore becomes

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/m_1 & 0 & 0 & -1/\sqrt{m_1 m_2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/\sqrt{m_1 m_2} & 0 & 0 & 1/m_2 \end{pmatrix} V''(\ell)\,. \tag{5.2.40}$$

The eigenvalues λ of $\mathbb{A}$ must satisfy the characteristic equation: $\det(\mathbb{A} - \lambda) = 0$, which in this case reads

$$\lambda^4 \left[\left(\frac{V''(\ell)}{m_1} - \lambda \right) \left(\frac{V''(\ell)}{m_2} - \lambda \right) - \frac{[V''(\ell)]^2}{m_1 m_2} \right] = \lambda^5 \left[\lambda - \frac{V''(\ell)}{\mu} \right] = 0\,, \tag{5.2.41}$$

where $\mu = m_1 m_2 / M$ is the reduced mass.

The matrix $\mathbb{A}$ has 5 zero eigenvalues with normalized eigenvectors

$$\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\0\\\sqrt{m_1/M}\\0\\0\\\sqrt{m_2/M}\end{pmatrix}, \tag{5.2.42}$$

and one nonzero eigenvector: the normalized eigenvector

$$\begin{pmatrix}0\\0\\-\sqrt{m_2/M}\\0\\0\\\sqrt{m_1/M}\end{pmatrix} \quad \text{has nonzero eigenvalue} \quad \lambda = V''(\ell)/\mu > 0\,. \tag{5.2.43}$$

This example shows how the general formalism works in a concrete example. For instance eq. (5.2.21) states $\ddot{\widehat{\mathbb{Y}}} = \mathbb{D}\,\widehat{\mathbb{Y}}$ and in a basis where $\mathbb{A}$ has been diagonalized we indeed have

$$\mathbb{D} = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&\omega^2 \end{pmatrix} \quad \text{where} \quad \omega^2 := \frac{V''(\ell)}{\mu}\,, \tag{5.2.44}$$

with eigenvectors

$$\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}. \tag{5.2.45}$$

Of these, the first five are the symmetry related zero eigenvectors, and only the nonzero eigenvector describes oscillatory motion when (5.2.21) is solved:

$$\widehat{\mathbb{Y}}(t) = \begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix} A\,\cos(\omega t + \delta)\,. \tag{5.2.46}$$

To write this in the original basis for which $\mathbb{A}$ is given in (5.2.40) we need the orthogonal rotation $\mathbb{O}$, which in practice is the matrix whose columns are the normalized eigenvectors of $\mathbb{A}$ given in (5.2.42) and (5.2.43), so

$$\mathbb{O} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&\sqrt{m_1/M}&0&0&-\sqrt{m_2/M}\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&\sqrt{m_2/M}&0&0&\sqrt{m_1/M} \end{pmatrix}, \tag{5.2.47}$$

as can be verified by direct matrix multiplication. In terms of this the eigenvectors (5.2.45) become $\widetilde{\mathbb{Y}} = \mathbb{O}\,\widehat{\mathbb{Y}}$ and so are given in the initial basis by

$$\widetilde{\mathbb{Y}} = \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\sqrt{m_1/M}\\0\\0\\\sqrt{m_2/M}\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-\sqrt{m_2/M}\\0\\0\\\sqrt{m_1/M}\end{pmatrix}, \tag{5.2.48}$$

in agreement with (5.2.42) and (5.2.43) (as they must).

Rescaling by the mass matrix then reveals the eigenvectors $\mathbb{Y} = \mathbb{M}^{-1/2}\,\widetilde{\mathbb{Y}}$ in the basis for which the evolution equation is $\mathbb{M}\ddot{\mathbb{Y}} = \mathbb{K}\mathbb{Y}$, where

$$\mathbb{Y} = \begin{pmatrix} 1/\sqrt{m_1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1/\sqrt{m_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{M} \\ 0 \\ 0 \\ 1/\sqrt{M} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/\sqrt{m_2} \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/\sqrt{m_2} \\ 0 \end{pmatrix}, \tag{5.2.49}$$

are all zero modes, while

$$\mathbb{Y}(t) = \begin{pmatrix} 0 \\ 0 \\ -\sqrt{\mu}/m_1 \\ 0 \\ 0 \\ \sqrt{\mu}/m_2 \end{pmatrix} A\cos(\omega t + \delta)\,, \tag{5.2.50}$$

corresponds to the single oscillatory normal mode (in agreement with the solutions seen earlier in §5.1). Notice that this solution satisfies $\mathbb{M}\,\mathbb{Y} = m_1\,\mathbf{y}_1 + m_2\,\mathbf{y}_2 = 0$ for the nonzero eigenvector, showing that the motion does not change the centre of mass position. During the oscillation the relative minus sign between the two entries of (5.2.50) show that the two atoms move with the opposite phase as they oscillate, and the mass-dependence shows how the amplitude of oscillation differs for each atom.

⋆ ⋆ ⋆

In order to see the superposition of more than one normal mode we need at lest three atoms, so we consider this example next.

Worked example: Triatomic Molecule

Consider a molecule involving three atoms all arranged along a single line. The central atom of the molecule has mass m_c and the two outer atoms have the same mass $m_\pm = m$ and potential energy as a function of their distance from the central atom and so share the same equilibrium distance ℓ from it (see the illustration in Fig. 26).

Let us choose the z axis parallel to the line connecting the three atoms in their equilibrium positions with the origin chosen to sit at the centre of mass, which by symmetry is also the position of the middle atom, so

$$\overline{\mathbf{s}}_\pm = \pm\ell\,\mathbf{e}_z \quad \text{and} \quad \overline{\mathbf{s}}_c = 0\,, \tag{5.2.51}$$

where (as usual) $M = m_+ + m_- + m_c = 2m + m_c$ is the total mass of the molecule. Notice $m_-\overline{\mathbf{s}}_- + m_+\overline{\mathbf{s}}_+ + m_c\overline{\mathbf{s}}_c = m(\overline{\mathbf{s}}_- + \overline{\mathbf{s}}_+) + m_c\overline{\mathbf{s}}_c = 0$ – as must happen in the centre of mass frame.

We take the interatomic forces to be derivable from a potential V that depends only on the relative distances between the atoms: $r_{c+} = |\mathbf{s}_+ - \mathbf{s}_c|$, $r_{c-} = |\mathbf{s}_- - \mathbf{s}_c|$ and $r_{+-} = |\mathbf{s}_+ - \mathbf{s}_-|$ (and so is translation and rotation invariant). This potential is assumed to have a local minimum for the positions given in (5.2.51), such as can be arranged if the atoms at the ends of the molecule are attracted to the central

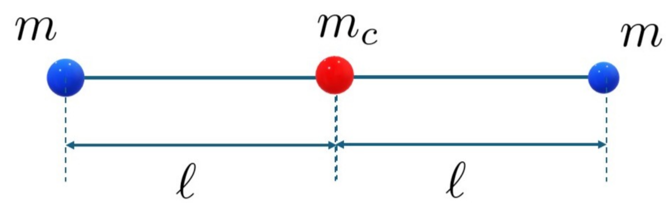

Figure 26. A sketch of the geometry of the collinear triatomic molecule whose vibrations are discussed in the text.

one through a potential, V_a, of the form drawn in Fig. (25), together with a repulsive interaction, V_r between the two end atoms. For such a potential the end atoms prefer to be a fixed distance ℓ away from the central one, and the repulsive nature of their direct interaction makes them prefer to remain on opposite sides of the central atom from one another.

In this case there are nine coordinates, y^A, with $A = \{a, i\}$ and $a = 1, 2, 3 = -, c, +$ and $i = x, y, z$,

$$\mathbb{Y} = \begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \\ y^5 \\ y^6 \\ y^7 \\ y^8 \\ y^9 \end{pmatrix} = \begin{pmatrix} y_{-x} \\ y_{-y} \\ y_{-z} \\ y_{cx} \\ y_{cy} \\ y_{cz} \\ y_{+x} \\ y_{+y} \\ y_{+z} \end{pmatrix}, \tag{5.2.52}$$

and we take the potential to have the approximate form

$$V = V_a(r_{c+}) + V_a(r_{c-}) + V_r(r_{+-}), \tag{5.2.53}$$

where once expanded about the equilibrium positions, $\mathbf{s}_a = \bar{\mathbf{s}}_a + \mathbf{y}_a$ the distances between atoms are

$$\begin{aligned} r_{c\pm} &= \sqrt{(y_{\pm x} - y_{cx})^2 + (y_{\pm y} - y_{cy})^2 + (y_{\pm z} - y_{cz} \mp \ell)^2} \\ \text{and} \quad r_{+-} &= \sqrt{(y_{+x} - y_{-x})^2 + (y_{+y} - y_{-y})^2 + (y_{+z} - y_{-z} - 2\ell)^2}\,. \end{aligned} \tag{5.2.54}$$

Differentiating gives

$$\begin{aligned} \frac{\partial V}{\partial y_{ci}} &= V_a'(r_{c+})\,\frac{\partial r_{c+}}{\partial y_{ci}} + V_a'(r_{c-})\,\frac{\partial r_{c-}}{\partial y_{ci}} \\ &= V_a'(r_{c+})\left(\frac{y_{ci} - y_{+i} + \ell\,\delta_{iz}}{r_{c+}}\right) + V_a'(r_{c-})\left(\frac{y_{ci} - y_{-i} - \ell\,\delta_{iz}}{r_{c-}}\right), \end{aligned} \tag{5.2.55}$$

which vanishes at $\mathbf{y}_a = 0$ so long as the equilibrium configuration satisfies $\overline{r}_{c+} = \overline{r}_{c-}$. Similarly

$$
\begin{aligned}
\frac{\partial V}{\partial y_{\pm i}} &= V_a'(r_{c+})\,\frac{\partial r_{c+}}{\partial y_{\pm i}} + V_a'(r_{c-})\,\frac{\partial r_{c-}}{\partial y_{\pm i}} + V_r'(r_{+-})\,\frac{\partial r_{+-}}{\partial y_{\pm i}} \\
&= V_a'(r_{c\pm})\left(\frac{y_{\pm i} - y_{ci} \mp \ell\,\delta_{iz}}{r_{c\pm}}\right) + V_r'(r_{+-})\left(\frac{y_{\pm i} - y_{\mp i} \mp 2\ell\,\delta_{iz}}{r_{+-}}\right), \qquad (5.2.56)
\end{aligned}
$$

which vanishes at $\mathbf{y}_a = 0$ if ℓ satisfies $V_a'(\ell) + V_r'(2\ell) = 0$.

Differentiating again is simplified if we assume the repulsive force is much weaker than the attractive force, so that ℓ approximately satisfies $V_a'(\ell) \simeq V_r'(\ell) \simeq 0$ relative to second derivatives and we can neglect $V_r''(\ell)$ relative to $V_a''(\ell)$. In this case the only non-negligible second derivatives at the minimum are

$$
\left(\frac{\partial^2 V}{\partial y_{\pm z}^2}\right)_0 \simeq V_a''(\ell) + V_r''(\ell)\,, \qquad \left(\frac{\partial^2 V}{\partial y_{+z}\partial y_{-z}}\right)_0 \simeq -V_r''(\ell)\,. \qquad (5.2.57)
$$

and

$$
\frac{1}{2}\left(\frac{\partial^2 V}{\partial y_{cz}^2}\right)_0 \simeq -\left(\frac{\partial^2 V}{\partial y_{cz}\partial y_{\pm z}}\right)_0 \simeq V_a''(\ell)\,. \qquad (5.2.58)
$$

Repeating the steps that led to eq. (5.2.31) gives the nine-by-nine matrix $k_{AB} = (\partial^2 V/\partial y^A \partial y^B)_{\mathbf{y}_a=0}$ evaluated at the equilibrium position ($\mathbf{y}_a = 0$), leading to

$$
k_{ai,bj} = \kappa_{ab}\,\sigma_{ij} \quad \text{where} \quad \kappa_{ab} = \begin{pmatrix} \kappa_{++} & \kappa_{+c} & \kappa_{+-} \\ \kappa_{c+} & \kappa_{cc} & \kappa_{c-} \\ \kappa_{-+} & \kappa_{-c} & \kappa_{--} \end{pmatrix} = \begin{pmatrix} k + \mathfrak{k} & -k & -\mathfrak{k} \\ -k & 2k & -k \\ -\mathfrak{k} & -k & k + \mathfrak{k} \end{pmatrix}, \qquad (5.2.59)
$$

and we define $\sigma_{ij} = \delta_{iz}\,\delta_{jz}$ as in (5.2.32) and

$$
k := V_a''(\ell) > 0 \qquad \text{and} \qquad \mathfrak{k} := V_r''(\ell)\,. \qquad (5.2.60)
$$

We remark in passing that the result (5.2.59) is different than what would have been found if we had followed the textbook *Goldstein* and defined the potential to have the approximate form

$$
\begin{aligned}
V &\simeq \tfrac{1}{2}k\,(\mathbf{s}_+ - \mathbf{s}_c - \ell\,\mathbf{e}_z)^2 + \tfrac{1}{2}k\,(\mathbf{s}_- - \mathbf{s}_c + \ell\,\mathbf{e}_z)^2 = \tfrac{1}{2}k\,(\mathbf{y}_+ - \mathbf{y}_c)^2 + \tfrac{1}{2}k\,(\mathbf{y}_- - \mathbf{y}_c)^2 \\
&= \tfrac{1}{2}k\,(\mathbf{y}_+^2 + \mathbf{y}_-^2) + k\,\mathbf{y}_c^2 - k\,\mathbf{y}_c\cdot(\mathbf{y}_+ + \mathbf{y}_-)\,. \qquad (5.2.61)
\end{aligned}
$$

In this case repeating the above steps gives the nine-by-nine matrix $k_{AB} = (\partial^2 V/\partial y^A \partial y^B)_{\mathbf{y}_a=0}$ evaluated at the equilibrium position ($\mathbf{y}_a = 0$) of the form

$$
k_{ai,bj} = \kappa_{ab}\,\delta_{ij} \quad \text{where} \quad \kappa_{ab} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}. \qquad (5.2.62)
$$

This differs from (5.2.59) by omitting the $\mathfrak{k}$ dependence, but more importantly in spin space this involves the matrix δ_{ij} rather than σ_{ij} and so it does not contain any rotational zero modes. This happens because the choice (5.2.61) does *not* give a rotation-invariant potential due to the appearance in it of the explicit direction $\ell\mathbf{e}_z$. By contrast, the choice (5.2.53) only acquires a dependence on $\ell\mathbf{e}_z$ once $\mathbf{s}_a$ is exchanged for $\mathbf{y}_a$.

The mass matrix in these coordinates similarly is $M_{ai,bj} = \mathcal{M}_{ab}\,\delta_{ij}$ (or $\mathbb{M} = \mathcal{M} \otimes I_3$) where I_3 is the 3×3 unit matrix and

$$\mathcal{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m \end{pmatrix} . \tag{5.2.63}$$

Written out in all of their glory, using this mass matrix and (5.2.59) gives the 9×9 matrices

$$\mathbb{K} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k+\mathfrak{k} & 0 & 0 & -k & 0 & 0 & -\mathfrak{k} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k & 0 & 0 & 2k & 0 & 0 & -k \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathfrak{k} & 0 & 0 & -k & 0 & 0 & k+\mathfrak{k} \end{pmatrix} \quad \text{and} \quad \mathbb{M} = \begin{pmatrix} m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m \end{pmatrix} . \tag{5.2.64}$$

The matrix $\mathbb{A} = \mathbb{M}^{-1/2}\,\mathbb{K}\,\mathbb{M}^{-1/2}$ therefore becomes

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (k+\mathfrak{k})/m & 0 & 0 & -k/\sqrt{mm_c} & 0 & 0 & -\mathfrak{k}/m \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k/\sqrt{mm_c} & 0 & 0 & 2k/m_c & 0 & 0 & -k/\sqrt{mm_c} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathfrak{k}/m & 0 & 0 & -k/\sqrt{mm_c} & 0 & 0 & (k+\mathfrak{k})/m \end{pmatrix} . \tag{5.2.65}$$

Solving the characteristic equation: $\det(\mathbb{A} - \omega^2) = 0$ for the eigenvalues ω^2 reveals seven independent zero eigenvectors and two nonzero eigenvalues (or characteristic frequencies):

$$\omega_1^2 = \frac{k + 2\mathfrak{k}}{m} \quad \text{and} \quad \omega_2^2 = \frac{k}{m}\left(1 + \frac{2m}{m_c}\right) . \tag{5.2.66}$$

Because the equilibrium configuration is a rotor (all atoms collinear) only five of the zero modes are consequences of translation and rotation symmetries. Under translation symmetry the positions of all atoms shift by a common amount, $\delta \mathbf{y}_a = \boldsymbol{\epsilon}$, corresponding to the following translational zero modes of the matrix $\mathbb{K}$:

$$\delta_{tx}\mathbb{Y} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \quad \delta_{ty}\mathbb{Y} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \delta_{tz}\mathbb{Y} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} , \tag{5.2.67}$$

Although the first two of these are also obvious zero eigenvectors of $\mathbb{A}$ the third one must be rescaled by powers of the mass due to the rescaling $\mathbb{Y} = \mathbb{M}^{-1/2}\widetilde{\mathbb{Y}}$ required to relate the original variable $\mathbb{Y}$ to the eigenvectors of $\mathbb{A}$. So the zero modes corresponding to translations in the z direction is given by

$$\delta_{tz}\widetilde{\mathbb{Y}} = \frac{1}{\sqrt{m_c + 2m}} \begin{pmatrix} 0 \\ 0 \\ \sqrt{m} \\ 0 \\ 0 \\ \sqrt{m_c} \\ 0 \\ 0 \\ \sqrt{m} \end{pmatrix}, \tag{5.2.68}$$

as is easily verified by direction multiplication by $\mathbb{A}$.

The other two zero modes associated with symmetries are those due to rigid rotations about the centre of mass of the molecule, for axes perpendicular to the z axis (which is the direction along which the atoms lie when in their equilibrium position). Since the central atom lies at the centre of mass it does not move under these rotations, and the equidistant positions of the outer two atoms makes them rotate in opposite directions. Explicitly, for an infinitesimal rotation about the x axis we have $\delta y_{-y} = \theta_x\,\ell$ and $\delta y_{+y} = -\theta_x\,\ell$ and an infinitesimal rotation about the y direction similarly gives $\delta y_{-x} = \theta_y\,\ell$ and $\delta y_{+x} = -\theta_y\,\ell$, so the corresponding zero eigenvectors of $\mathbb{K}$ (and $\mathbb{A}$) therefore are

$$\delta_{rx}\mathbb{Y} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \delta_{ry}\mathbb{Y} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}. \tag{5.2.69}$$

Inspection of $\mathbb{A}$ shows that the remaining non-symmetry zero modes involve transverse motion of the central atom in the x and y directions without also moving the outer ones:

$$\delta_{rx}\mathbb{Y} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \delta_{ry}\mathbb{Y} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.2.70}$$

These are only approximate zero modes since their eigenvalues only vanish because of the simplifying approximations made when evaluating $\mathbb{K}$, in particular the assumption that this matrix is dominated by $V_a''(\ell)$ and $V_r''(\ell)$. This assumption neglects the restoring force that acts transverse to the molecular axis when the central atom is 'plucked' away from its equilibrium position and allowed to oscillate.

The remaining eigenvectors correspond to the nonzero eigenvalues given in (5.2.66). The eigenvector corresponding to the first nonzero eigenvalue is given by

$$\mathbb{Y}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} A\cos(\omega_1 t + \delta) \quad \text{for the eigenvalue} \quad \omega_1^2 = \frac{k + 2\mathfrak{k}}{m} . \tag{5.2.71}$$

This involves only the outer atoms oscillating with frequency ω_1 in opposite directions in the z direction while the central atom does not move (as it must in order for the molecule not to carry net momentum). Because only the outer atoms move the frequency depends only on the mass m of the outer atoms.

The other eigenvector with nonzero eigenfrequency for the matrix $\mathbb{A}$ is

$$\widetilde{\mathbb{Y}}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -2\sqrt{m/m_c} \\ 0 \\ 0 \\ 1 \end{pmatrix} \widetilde{A}\cos(\omega_2 t + \delta) \quad \text{for the eigenvalue} \quad \omega_2^2 = \frac{k}{m}\left(1 + \frac{2m}{m_c}\right) . \tag{5.2.72}$$

More useful is to have this oscillation in terms of the original variables $\mathbb{Y}$, keeping in mind the relation $\mathbb{Y} = \mathbb{M}^{-1/2}\widetilde{\mathbb{Y}}$. Absorbing a factor of $\sqrt{m}$ by defining $\widetilde{A} = \sqrt{m}A$, this can be written

$$\mathbb{Y}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -2m/m_c \\ 0 \\ 0 \\ 1 \end{pmatrix} A\cos(\omega_2 t + \delta) \quad \text{for the eigenvalue} \quad \omega_2^2 = \frac{k}{m}\left(1 + \frac{2m}{m_c}\right) . \tag{5.2.73}$$

This corresponds to both of the outer atoms oscillating in the same direction with frequency ω_2 while the central atom simultaneously oscillates in the other direction with the amplitude required to ensure that its instantaneous momentum precisely cancels the momenta of the outer two atoms. That is, the momenta along the z direction for of each type of atom performing this oscillation are

$$m\dot{y}_{-z} = m\dot{y}_{+z} = -mA\omega_2 \sin(\omega_2 t + \delta) \quad \text{and} \quad m_c \dot{y}_{cz} = +2mA\omega_2 \sin(\omega_2 t + \delta)\,, \tag{5.2.74}$$

and so $m\dot{y}_{-z} + m\dot{y}_{+z} + m_c\dot{y}_{cz} = 0$.

All told our three-atom triatomic molecule has 9 degrees of freedom of which 5 describe rigid body translations and rotations. The remaining 4 degrees of freedom describe 4 independent normal modes of oscillation, each with its own characteristic frequency, for small displacements of the constituent atoms about their equilibrium positions. The two modes for which the central atom moves transverse to the molecule's axis have systematically smaller frequencies compared to oscillations for which the atoms move along the line connecting the equilibrium positions of the atoms. A general vibration of the molecule can be written as a linear combination of these four normal modes.

⋆ ⋆ ⋆

One can continue in this way, adding more and more atoms to obtain larger and larger macroscopic bodies. In general each atom adds another 3 normal modes (one for each of the three spatial directions). The number of zero modes coming from spatial symmetries in general remains fixed at six (three translations and three rotations – or possibly only two independent rotations if all the atoms are collinear).

5.3 Continuum of Oscillators

The number of normal modes arising in this way for an oscillating macroscopic body can be truly enormous once the number N of atoms becomes sizable. The motion can sometimes nonetheless be fairly simple and acquire a universal behaviour in some circumstances – most notably when the system is only observed over distance scales that are very large compared with the typical interatomic spacings and over time scales that are long compared with the periods, $T = 2\pi/\omega$, of the typical characteristic frequencies. This section is devoted to what happens in this limit and why simplicity can emerge.

As we saw in the previous section, normal modes often involve adjacent atoms moving in very different ways. For instance, for the triatomic molecule one of a pair of adjacent atoms could be completely motionless while its immediate neighbor oscillates like crazy. These kinds of oscillations can be very hard to detect if one can only resolve distance scales that are very large compared to the interatomic spacing. If only comparatively large distances can be resolved then the only oscillations that can be detected in practice are those that involve a macroscopic number of atoms moving together in a very similar way.

It also turns out that these are the normal modes that typically involve the lowest frequencies: for N oscillators coupled only to their nearest neighbours the lowest-frequency normal modes tend to be the ones where all of the particles are moving as similarly to one another as possible, with none moving completely out of phase with the others. This kind of low-frequency coherent motion is often universal in its character (in the sense that it does not depend in detail on the properties of the underlying atoms).

There is a good reason why these low frequency modes of oscillation become universal. We have seen that if *all* the atoms of an object move with *exactly* the same speed in the same direction then this is not really an oscillation at all. It is instead a rigid body translation. It can cost arbitrarily little energy if the motion is slow enough because there is no potential energy cost if the interatomic interactions are translation invariant. But if there are a great many atoms present then some oscillations can involve large groups of atoms moving together with velocities that are *almost* but not exactly the same. If the change in velocity from atom to atom is sufficiently small the energy cost of this motion can also be made arbitrarily small because it locally is almost a symmetry transformation. (It becomes an honest-to-God symmetry in the limit that all atoms move with exactly the same speed and direction.) It is the intimate relationship between this type of motion and symmetries that makes the resulting behaviour universal.

5.3.1 The oscillating string

To explore these ideas more concretely consider first the simple system of a long line (or a long and comparatively narrow bundle) of atoms that are too closely spaced to distinguish individual atoms, but which are free to move transverse to the line's initial direction.

We wish to write down the Lagrangian for such a string of atoms in order to obtain its equations of motion. To this end we start with the kinetic energy of the motion. Suppose then that we have a narrow string or wire of cross sectional area $\mathcal{A}$ whose typical length $a \sim \sqrt{\mathcal{A}}$ is both much smaller than the resolution of our position measurements and much larger than the spacing ℓ between atoms.

Because we cannot resolve its width the wire will look like a one-dimensional object, with length but no width, and so its position is specified by giving the curve $\mathbf{z}(u)$ that gives the wire's position as a function of time. Here u is any convenient parameter (we later use arc-length measured along the wire for this purpose, but that is not required) that labels a particular 'point' along the wire's length. It is possible that the wire's cross-sectional area $\mathcal{A} = \mathcal{A}(u)$ varies along the wire.

We know that in secret every point on the curve $\mathbf{z}(u)$ contains many atoms all more closely packed than we can resolve. Consider a small segment of this wire contained in an interval $\mathrm{d}u$ about some value for u, chosen so that the length, $\mathrm{d}s$, of this interval is at the limit of resolution of our measurements. That is, $\mathrm{d}s$ is much shorter than any process we hope to actually measure, but is also much larger than the interatomic spacing within the wire: $\mathrm{d}s \gg a \gg \ell$.

Given an explicit parameterization $\mathbf{z}(u) = \{x(u), y(u), z(u)\}$ of the wire's position, the vector $\partial\mathbf{z}/\partial u$ is tangent to the wire. The the length of the wire lying in the interval $\mathrm{d}u$ is

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2} = \sqrt{\mathrm{d}\mathbf{z} \cdot \mathrm{d}\mathbf{z}} = \sqrt{\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}u} \cdot \frac{\mathrm{d}\mathbf{z}}{\mathrm{d}u}}\, \mathrm{d}u\,, \tag{5.3.1}$$

and so $\mathrm{d}s/\mathrm{d}u = \sqrt{(\mathrm{d}\mathbf{z}/\mathrm{d}u)\cdot(\mathrm{d}\mathbf{z}/\mathrm{d}u)}$. If u is a reliable label of points along the wire it increases monotonically as one moves along the wire and so we can assume $\mathrm{d}s/\mathrm{d}u > 0$ is strictly positive everywhere so the map $u \to s$ is one-to-one. It is often convenient to use s itself as the parameter labelling points along the wire. In this case (5.3.1) shows that

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}s}\cdot\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}s} = 1\,, \tag{5.3.2}$$

(so the tangent vector is in this case automatically normalized).

If we could only resolve it, the volume of this segment of the wire would be

$$\mathrm{d}V = \mathcal{A}(s)\,\mathrm{d}s = \mathcal{A}(u)\left(\frac{\mathrm{d}s}{\mathrm{d}u}\right)\mathrm{d}u\,, \tag{5.3.3}$$

and so the total mass within this segment is $\mathrm{d}m = \rho(s,t)\,\mathrm{d}V$ where ρ is the mass-per-unit-volume of the atoms in the wire. In terms of the microscopic atoms

$$\mathrm{d}m = \rho(s)\,\mathcal{A}(s)\,\mathrm{d}s = \sum_{a\in\mathrm{d}V} m_a\,, \tag{5.3.4}$$

where m_a is the mass of atom 'a' and the sum runs only over those atoms that lie within the volume element $\mathrm{d}V$. If we cannot resolve the width of the wire then what mattters is the wire's mass-per-unit-length:

$$\sigma(s) := \frac{\mathrm{d}m}{\mathrm{d}s} = \rho(s)\,\mathcal{A}(s)\,. \tag{5.3.5}$$

Next we restrict our attention to atoms that are all mostly moving in the same direction, doing so by assuming that all of the atoms in the volume $\mathrm{d}V$ have the same velocity, $\mathbf{v}$, to within the tolerance of our measurements, but allowing for the possibility that this common velocity varies slowly with s, $\mathbf{v} = \mathbf{v}(s)$, as we move down the wire. Having the entire segment move together at low energies is a natural consequence of the interatomic forces $\mathbf{F}_{ab}$ that act between the atoms (and provide them with a preferred equilibrium distance from all of their neighbours). In particular, it provides a strong repulsive part that prevents atoms being pushed into one another.

Under these circumstances the motion of a segment of the wire is transverse to the direction along the wire. For instance, if the wire is initially laid along the z axis then the repulsion of atoms requires a wire segment to move only in the x-y plane, so $\mathbf{v} = v_x\,\mathbf{e}_x + v_y\,\mathbf{e}_y$. In practice this means that the motion of atoms within a short segment of length $\mathrm{d}s$ simply changes the shape of the wire, so the velocity gives the rate of change of the curve that defines the wire's position:

$$\mathbf{v}(s,t) = \partial_t\mathbf{z}(s,t) =: \dot{\mathbf{z}}(s,t)\,, \tag{5.3.6}$$

and

$$\mathbf{v}(s,t)\cdot\partial_s\mathbf{z}(s,t) = 0\,. \tag{5.3.7}$$

The kinetic energy due to this motion of the volume $\mathrm{d}V$ then is

$$\mathrm{d}K = \tfrac{1}{2}\,\mathrm{d}m\,\mathbf{v}^2 = \tfrac{1}{2}\,\rho\,\mathbf{v}^2\,\mathrm{d}V = \tfrac{1}{2}\,\rho\,\mathbf{v}^2\,\mathcal{A}\,\mathrm{d}s = \tfrac{1}{2}\,\sigma\,\mathbf{v}^2\,\mathrm{d}s\,. \tag{5.3.8}$$

For a material wire there is also a potential energy cost to moving one part of the wire relative to its neighbours. This also comes from the interatomic forces $\mathbf{F}_{ab}$ that act between the atoms. The attractive part of the interactomic forces resists the stretching of the wire since it imposes an energy cost on changing the wire's length. If the curve defining the wire changes from $\overline{\mathbf{z}}(u)$ to $\mathbf{z}(u) = \overline{\mathbf{z}}(u) + \mathbf{y}(u)$, with $\mathbf{y}$ perpendicular to the tangent $\mathrm{d}\overline{\mathbf{z}}/\mathrm{d}u$ and we parameterize using the arc-length, $\bar{s}$, for the original curve $\overline{\mathbf{z}}(u)$ then (5.3.1) tells us that $|\mathrm{d}\overline{\mathbf{z}}/\mathrm{d}\bar{s}| = 1$ and predicts the change in the length of a segment initially of length $\mathrm{d}\bar{s}$ to be

$$\delta\mathrm{d}s = \left[\sqrt{1 + \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\bar{s}}\cdot\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\bar{s}}} - 1\right]\mathrm{d}\bar{s} \simeq \tfrac{1}{2}\left(\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\bar{s}}\right)^2\,\mathrm{d}\bar{s}\,, \tag{5.3.9}$$

where the cross term vanishes[19] in the first equality because $(\mathrm{d}\overline{\mathbf{z}}/\mathrm{d}\bar{s})\cdot(\mathrm{d}\mathbf{y}/\mathrm{d}\bar{s}) = 0$ and the approximate equality assumes small displacements, so $|\mathrm{d}\mathbf{y}/\mathrm{d}\bar{s}| \ll 1$.

The potential energy cost of such a deformation is the work done against the total restoring interatomic force by this change of length. Once aggregated over the interior of the wire this restoring force is called the wire's *tension*: $T(s) > 0$, and is regarded as being one of its macroscopic properties, like its mass-per-unit-length $\sigma(s)$. The potential energy change due to a small displacement of the wire therefore is

$$\mathrm{d}V = T(s)\,\delta\mathrm{d}s \simeq \tfrac{1}{2}\,T(\bar{s})\left(\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\bar{s}}\right)^2\,\mathrm{d}\bar{s} \simeq \tfrac{1}{2}\,T(s)\left(\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}s}\right)^2\,\mathrm{d}s\,, \tag{5.3.10}$$

where the last equality uses that the difference between s and $\bar{s}$ is higher order in $\mathrm{d}\mathbf{y}/\mathrm{d}s$.

This expression for the potential energy vanishes if $\partial\mathbf{y}/\partial s = 0$ because the limit of a constant displacement of the whole wire is a rigid-body motion for which there is no cost in potential energy. Working to quadratic order in $\partial\mathbf{y}/\partial s$ in (5.3.10) is the analog in this instance of using the harmonic approximation when exploring the potential energy of oscillations of individual atoms around their equilibrium positions.

We now have the tools to write down the Lagrangian for the aggregate motion of small deviations of atoms aligned along a narrow wire, at least in the low-energy limit appropriate for asking questions over time scales much longer than the characteristic frequencies of the generic atomic normal modes of vibration (so that oscillatory motion over short unresolved distances can be ignored). The Lagrangian is as usual the difference between the total kinetic and potential energies which, using (5.3.8) and (5.3.10), becomes

$$L = K - V = \int_{s_0}^{s_1}\mathrm{d}s\,\left(\frac{\mathrm{d}K}{\mathrm{d}s} - \frac{\mathrm{d}V}{\mathrm{d}s}\right) = \int_{s_0}^{s_1}\mathrm{d}s\,\left[\tfrac{1}{2}\,\sigma\,(\partial_t\mathbf{y})^2 - \tfrac{1}{2}\,T\,(\partial_s\mathbf{y})^2\right]. \tag{5.3.11}$$

[19] A more systematic treatment including when the cross term cannot be neglected is given in §9.

As usual, eq. (5.3.11) is to be integrated over time to obtain the action, so

$$S[\mathbf{y}(s,t)] = \int_{t_0}^{t_1} \mathrm{d}t\, L = \int_{t_0}^{t_1} \mathrm{d}t \int_{s_0}^{s_1} \mathrm{d}s\, \Big[\tfrac{1}{2}\,\sigma\,(\partial_t \mathbf{y})^2 - \tfrac{1}{2}\,T\,(\partial_s \mathbf{y})^2\Big]\,. \tag{5.3.12}$$

This is now a functional of a configuration $\mathbf{y}(s,t)$ that is a function of *two* variables rather than just time. The equations of motion are obtained in precisely the same way, though, by asking which configurations give $\delta S = 0$ stationary under arbitrary variations $\delta\mathbf{y}(s,t)$.

In this formulation the two parameters s and t are conceptually treated very differently. The parameter s and the vector index $i = x, y, z$ in $y_i(s,t)$ are collectively treated effectively the same as the label A in the generalized coordinate $q^A(t)$ introduced in §2.2. The main difference is that s is a continuous variable rather than a discrete one, so sums over A turn into sums over i and integrals over s.

5.3.2 The wave equation

To derive the equations of motion we take the difference of the action evaluated at $\mathbf{y}(s,t) + \delta\mathbf{y}(s,t)$ and its value evaluated at $\mathbf{y}(s,t)$, keeping only linear terms in $\delta\mathbf{y}(s,t)$ and its derivatives. Taylor expanding the integrand of (5.3.12) to linear order gives

$$\begin{aligned} \delta S &= \int_{t_0}^{t_1} \mathrm{d}t \int_{s_0}^{s_1} \mathrm{d}s\, \Big[\sigma\, \partial_t \mathbf{y} \cdot \partial_t(\delta\mathbf{y}) - T\, \partial_s \mathbf{y} \cdot \partial_s(\delta\mathbf{y})\Big] \\ &= \int_{s_0}^{s_1} \mathrm{d}s\, \Big[\sigma\, \partial_t \mathbf{y} \cdot \delta\mathbf{y}\Big]_{t_0}^{t_1} - \int_{t_0}^{t_1} \mathrm{d}t\, \Big[T\, \partial_s \mathbf{y} \cdot \delta\mathbf{y}\Big]_{s_0}^{s_1} \\ &\qquad + \int_{t_0}^{t_1} \mathrm{d}t \int_{s_0}^{s_1} \mathrm{d}s\, \Big[-\partial_t\Big(\sigma\, \partial_t \mathbf{y}\Big) + \partial_s\Big(T\, \partial_s \mathbf{y}\Big)\Big] \cdot \delta\mathbf{y}(s,t)\,. \end{aligned} \tag{5.3.13}$$

The goal is to require δS vanish for *arbitrary* variations $\delta\mathbf{y}(s,t)$. We first demand $\delta S = 0$ for those $\delta\mathbf{y}(s,t)$ that vanish at the boundaries $t = t_0$ and t_1 and $s = s_0$ and s_1. In this case only the final integral in (5.3.13) contributes and so requiring $\delta S = 0$ for arbitrary $\delta\mathbf{y}(s,t)$ implies

$$-\partial_t\Big(\sigma\, \partial_t \mathbf{y}\Big) + \partial_s\Big(T\, \partial_s \mathbf{y}\Big) = 0\,. \tag{5.3.14}$$

If it is also true that the action is to be stationary when $\delta\mathbf{y} \neq 0$ at the endpoints then one gets the additional conditions

$$\begin{aligned} \partial_t \mathbf{y} &= 0 \quad \text{for all } s \text{ when } t = t_0 \text{ and } t_1 \\ \text{and} \quad \partial_s \mathbf{y} &= 0 \quad \text{for all } t \text{ when } s = s_0 \text{ and } s_1. \end{aligned} \tag{5.3.15}$$

For cases where both T and σ are independent of s and t the field equation (5.3.14) becomes

$$-\partial_t^2 \mathbf{y} + c_s^2\, \partial_s^2 \mathbf{y} = 0\,, \quad \text{where} \quad c_s := \sqrt{\frac{T}{\sigma}}\,. \tag{5.3.16}$$

The quantity c_s has the dimension of velocity. Eq. (5.3.16) is a famous equation and is called the *wave equation*, since its solutions describe wave propagation. As we see below c_s turns out to be the wave speed.

Eq. (5.3.16) is simple enough that it can be solved exactly (provided c_s is independent of s and t). The general solution can be found by changing variables from s and t to $u = s - c_s t$ and $v = s + c_s t$. In terms of these variables (5.3.16) becomes

$$\frac{\partial^2 \mathbf{y}}{\partial u \, \partial v} = 0 \, . \tag{5.3.17}$$

This integrates to give the general solution

$$\mathbf{y}(u, v) = \mathbf{f}_1(u) + \mathbf{f}_2(v) \, , \tag{5.3.18}$$

where $\mathbf{f}_1(u)$ and $\mathbf{f}_2(v)$ are arbitrary vector functions, subject only to the conditions that they (like $\mathbf{y}$) are perpendicular to the background configuration $\overline{\mathbf{z}}$. In terms of the original variables this becomes

$$\mathbf{y}(s, t) = \mathbf{f}_1(s - c_s t) + \mathbf{f}_2(s + c_s t) \, . \tag{5.3.19}$$

Eq. (5.3.19) provides the key to interpreting (5.3.16) as a wave equation. This is because for any profile $\mathbf{f}_1(x)$ the function $\mathbf{f}(s - c_s t)$ describes the translation of the profile to larger and larger values of s as t changes, where the the translation occurs at speed c_s (see Fig. 27). A similar story goes through for $\mathbf{f}_2(s + c_s t)$ though this translates the profile to smaller and smaller values of s (again with speed c_s) as t increases.

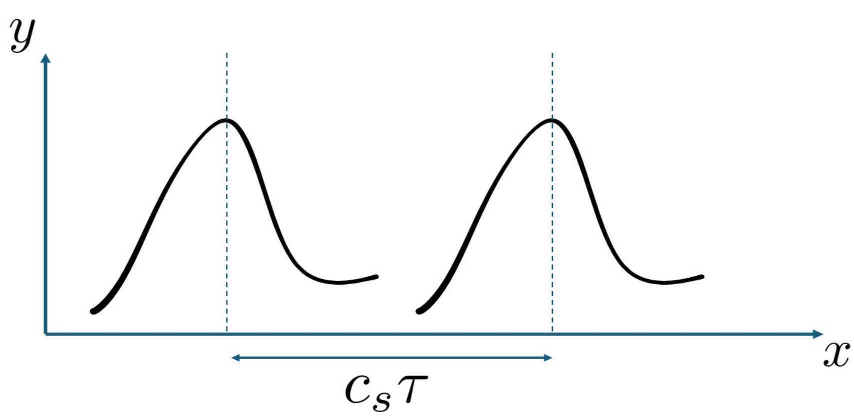

Figure 27. The evolution of a profile $y = f(x \pm c_s t)$ showing how the fixed profile translates through a distance $\Delta x = \mp c_s \tau$ after time evolves through an interval $\Delta t = \tau$. (For instance if $f(x)$ has a maximum at $x = 0$ when $t = 0$ then when $t = \tau$ the maximum still occurs when the argument of f vanishes, but for $f(x - c_s \tau)$ this happens when $x = c_s \tau$.)

The general solution to (5.3.16) is clearly the superposition of a left-moving and a right-moving wave, whose specific profiles can be determined by the initial conditions:

$$\mathbf{y}(s,0) = \mathbf{f}_1(s) + \mathbf{f}_2(s) \quad \text{and} \quad \partial_t \mathbf{y}(s,0) = c_s \Big[-\mathbf{f}_1'(s) + \mathbf{f}_2'(s) \Big] \,. \tag{5.3.20}$$

For instance if $\mathbf{y}(s,0) = \mathbf{h}(s)$ is specified and the wire is initially at rest, $\partial_t \mathbf{y}(s,0) = 0$, then we learn $\mathbf{f}_1'(s) = \mathbf{f}_2'(s)$ and so $\mathbf{f}_1(s) = \mathbf{f}_2(s) + C$ for some constant C, and so $\mathbf{f}_2(s) = \frac{1}{2}[\mathbf{h}(s) - C]$. In this case the solution is

$$\mathbf{y}(s,t) = \mathbf{f}_2(s - c_s t) + \mathbf{f}_2(s + c_s t) + C = \tfrac{1}{2} \Big[\mathbf{h}(s - c_s t) + \mathbf{h}(s + c_s t) \Big] \,. \tag{5.3.21}$$

5.3.3 Normal Modes (again)

It is also possible to seek normal mode solutions to (5.3.16), which can be defined to have a simple time-dependence $\mathbf{y}(s,t) = \mathbf{u}(s)\, e^{-i\omega t}$. In this case (5.3.16) states that the function $\mathbf{u}(s)$ must satisfy

$$\partial_s^2 \mathbf{u} + k^2 \mathbf{u} = 0 \quad \text{where} \quad k := \omega / c_s \,. \tag{5.3.22}$$

This is solved by

$$\mathbf{u} = \mathbf{a}_1\, e^{iks} + \mathbf{a}_2\, e^{-iks} \tag{5.3.23}$$

where k (or equivalently ω) and the constant vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are chosen to satisfy any spatial boundary conditions. For instance if $\mathbf{y}(s,t) = 0$ for $s = 0$ and $s = L$ then $\mathbf{u}_n(s) = \mathbf{a}_n \sin(k_n s)$ where $k_n = n\pi/L$ for $n = 1, 2, 3, \cdots$ a positive integer, leading to the characteristic frequencies $\omega_n = n\pi c_s / L$. The remaining constants $\mathbf{a}_n$ can then be chosen by imposing a convenient normalization condition on the normal modes in question.

Just as was true when all of the atoms can be resolved, a general solution to the wave equation (together with an appropriate set of boundary conditions) can be written as a linear combination of normal modes, each of which has a characteristic frequency. Furthermore, the relative amplitude of oscillation at each point of the string is fixed once the specific normal mode is chosen, exactly like what happens when the motion of each atom can be resolved. The particular combination of normal modes that arises in any particular physical problem is obtained by matching to the initial conditions, $y(s,0)$ and $\partial_t y(s,0)$.

In the continuum case we study here this normal-mode decomposition just turns out to be a Fourier series representation of wave motion. To see why this is true it is useful to examine a particular case in more detail. To this end, suppose the displacement satisfies the boundary condition that it vanishes at the edges of an interval at all times: $\mathbf{y}(0,t) = \mathbf{y}(L,t) = 0$. Consistency requires any initial conditions also to satisfy this boundary condition.

For instance, if $\mathbf{y}(s,0) = \mathbf{h}(s)$ and $\partial_t \mathbf{y}(s,0) = 0$ (as used above) then consistency requires $\mathbf{h}(0) = \mathbf{h}(L) = 0$). In this case it is always possible to find coefficients $\mathbf{h}_n$ such that

$$\mathbf{h}(s) = \sum_{n=1}^{\infty} \mathbf{h}_n \sin\left(\frac{n\pi s}{L} \right) \,. \tag{5.3.24}$$

The required coefficients can be found by using the identity

$$\int_0^L ds\ \sin\left(\frac{n\pi s}{L}\right)\ \sin\left(\frac{m\pi s}{L}\right) = \frac{L}{2}\,\delta_{mn}\,, \tag{5.3.25}$$

since it shows that $\mathbf{h}_m$ can be found by multiplying (5.3.24) through by $\sin(m\pi s/L)$ and integrating:

$$\mathbf{h}_n = \frac{2}{L}\int_0^L ds\ \mathbf{h}(s)\ \sin\left(\frac{n\pi s}{L}\right)\ . \tag{5.3.26}$$

Using (5.3.24) in (5.3.27) then gives the solution as an expansion of normal modes, of the form

$$\mathbf{y}(s,t) = \frac{1}{2}\sum_{n=1}^{\infty}\mathbf{h}_n\left\{\sin\left[\frac{n\pi}{L}(s-c_st)\right] + \sin\left[\frac{n\pi}{L}(s+c_st)\right]\right\}\ , \tag{5.3.27}$$

with $\mathbf{h}_n$ given in terms of the initial condition $\mathbf{y}(s,0) = \mathbf{h}(s)$ by (5.3.26).

5.4 Spin Waves*

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5.4.1 Antiferromagnets

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5.4.2 Ferromagnets

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6 Relativistic Systems

The two great revolutions in physics during the early twentieth century – relativity and quantum mechanics – are widely perceived as both requiring the abandonment of Newton's formulation of nature's laws. Although this is certainly true for quantum physics the same is not strictly true for relativity provided one takes the Lagrangian formulation of Newton's law's as primary, since there is a Lagrangian formulation for classical relativistic motion.

The main thing to change is the specific form for how forces and energies depend on position and velocity, which is more about the choice of the Lagrangian used in a particular situation rather than a complete abandonment of the formalism. This section seeks to clarify this by showing how special relativity can be embedded into the Lagrangian framework of classical mechanics.

6.1 Poincaré *vs* Galilean Invariance

From the point of view of Lagrangian mechanics the main difference between special relativity and Newtonian mechanics is the form that is assumed for spacetime symmetries. As listed in §1.6 the spacetime symmetries assumed in Newtonian mechanics come in three types:

- Translations in space and time: $\mathbf{r} \to \mathbf{r} + \mathbf{c}$ and $t \to t + \tau$;
- Rotations in space: $\mathbf{r} \to R\,\mathbf{r}$, where $R \in O(3)$ is a 3×3 rotation matrix, and
- Galilean boosts: $\mathbf{r} \to \mathbf{r} + \mathbf{u}\,t$ (that relates inertial observers who move with velocities that differ by the constant velocity $\mathbf{u}$).

Although translations and rotations have played an important role in discussions to this point, invariance of the laws of physics under Galilean boosts has been more of an academic observation.

Translation and rotation invariance do not change at all when we pass to special relativity; it is the invariance under Galilean boosts that changes. It might seem odd that this is possible since the derivation of the transformation $\mathbf{r} \to \mathbf{r} + \mathbf{u}\,t$ given in §1.6 seems to rely only on the properties of vector addition and so is not that complicated. As it turns out, the 'mistake' in this derivation is to assume that time is universal for all inertial observers, as we see more explicitly in §6.1.2 below.

Before exploring how relativity differs it is worth first describing in more detail how Galilean invariance restricts the choices we have made to this point.

6.1.1 Galilean constraints on L

Consider first the Lagrangian $L(\mathbf{r}, \dot{\mathbf{r}}, t)$ describing the motion of a single particle, for which the sole degree of freedom in the position $\mathbf{r}(t)$. Invariance under spatial and time translations implies L cannot depend on $\mathbf{r}$ undifferentiated and it cannot depend explicitly on t, and so $L = L(\dot{\mathbf{r}})$. Rotation invariance then further restricts $L = L(v^2)$ to be a function of $v^2 = \dot{\mathbf{r}}^2$ since this is the only rotational scalar that can be built using only $\dot{\mathbf{r}}$.

The functional form for $L(v^2)$ is further constrained by requiring physics to be invariant under Galilean boosts, for which $\dot{\mathbf{r}} \to \dot{\mathbf{r}} + \mathbf{u}$ for arbitrary constant $\mathbf{u}$, and so $v^2 \to v^2 + 2\mathbf{u}\cdot\dot{\mathbf{r}} + u^2$. If we were to demand L be invariant under this transformation then we would learn the absurd result that L must be independent of v^2 and so just be a constant.

Happily this is too strong a condition because we must only demand the action be invariant and so L can change under a Galilean boost so long as it changes by an additive constant or a total time derivative (or both). But this is still a restrictive condition because it implies that $L(v^2)$ must be linear in v^2. Defining the coefficient to be $\frac{1}{2}m$ this means the unique result for the Galilean-invariant Lagrangian for a single particle is

$$L = \tfrac{1}{2} m v^2 \quad \text{for which} \quad L \to L + \tfrac{1}{2} m u^2 + \frac{\mathrm{d}}{\mathrm{d}t}\Big(m\mathbf{u} \cdot \mathbf{r} \Big)\,. \tag{6.1.1}$$

We see from this argument that what might have seemed an arbitrary choice for the kinetic energy of a single particle is really the only choice that is consistent with Galilean invariance. Turning this argument around, we see from the above Lagrangian that Galilean invariance requires the equations of motion for an isolated particle to have the form $m\,\ddot{\mathbf{r}} = 0$, which indeed is Newton's first law (in the absence of forces a particle moves in a straight line at constant speed).

More possibilities are of course possible once more than one particle is present. For instance, for two particles invariance under time translations again requires L not to depend separately on t but invariance under spatial translations is now possible if $\mathbf{r}_1$ and $\mathbf{r}_2$ appear undifferentiated only through the difference $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Invariance under rotations requires $\mathbf{r}$, $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ to appear only through the rotation invariants that can be built from these vectors: $\mathbf{r}^2$, $\dot{\mathbf{r}}_1^2$, $\dot{\mathbf{r}}_2^2$, $\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_2$, $\mathbf{r} \cdot \dot{\mathbf{r}}_1$ and $\mathbf{r} \cdot \dot{\mathbf{r}}_2$. What remains is to impose invariance under Galilean boosts.

There is no loss of generality in writing $L = K - U$ where $K = \frac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{r}}_2^2 + m_{12}\, \dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_2$ is an arbitrary quadratic function of the velocities and U is everything else (and so in principle differs from the scalar potential in that it can depend on velocities). Although K is not invariant under Galilean transformations it does transform into a constant plus a total time derivative, precisely as was the case for a single particle. The function U can depend on both $\mathbf{r}$ and the velocities $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ but the result can only be Galilean invariant under $\dot{\mathbf{r}}_a \to \dot{\mathbf{r}}_a + \mathbf{u}$ if the velocities appear in U only through the difference $\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2 = \dot{\mathbf{r}}$. We are led to the conclusion that $U = U(\mathbf{r}^2, \dot{\mathbf{r}}^2, \mathbf{r} \cdot \dot{\mathbf{r}})$ can be a function of three independent invariants. In spherical polar coordinates (r, θ, ϕ) we have $\mathbf{r}^2 = r^2$, $\dot{\mathbf{r}}^2 = \dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2)$ and $\mathbf{r} \cdot \dot{\mathbf{r}} = r\dot{r}$, so $U = U(r, \dot{r}, \dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2)$.

We see that Galilean invariance (like invariance under rotations and translations) does constrain the form allowed for L, with the strongest constraints arising in the case where only a single particle is present.

6.1.2 Lorentz invariance

In special relativity the logic for L is similar: we choose the Lagrangian of any system from the subset of those that are invariant under translations, rotations and boosts between inertial observers moving at different speeds. The only change is that the explicit form for the transformation of boosts is no longer given by

$$\mathbf{r} \to \mathbf{r} + \mathbf{v}\, t \tag{6.1.2}$$

when the two inertial observers' velocities differ by $\mathbf{v}$.

Historically, the need for this change was driven by the discovery and success of Maxwell's theory of electromagnetism. According to this theory the equations of motion for electric and magnetic fields are *not* invariant under (6.1.2). A particular manifestation of this fact is in the theory's prediction of electromagnetic waves: in the absence of electric charges and currents

Maxwell's equations imply $\mathbf{E}$ satisfies

$$-\frac{\partial^2 \mathbf{E}}{\partial t^2} + c^2 \nabla^2 \mathbf{E} = 0 \tag{6.1.3}$$

and similarly for $\mathbf{B}$, where c is a quantity that is calculable in terms of the vacuum's dielectric permeability and permittivity, taking the numerical value $c = 299,792,458$ m/s, agreeing well with the speed of light in vacuum. Indeed (6.1.3) is called the wave equation – compare it with (5.3.16) for waves along a wire – and has wave solutions whose propagation speed is c.

The ability to calculate the speed of light in this way starting from equations of electricity and magnetism revealed that light was a manifestation of oscillating electric and magnetic fields having a particular range of wavelengths, and it predicted the existence of a whole spectrum of waves at other wavelengths on which our subsequent technology has since come to rely. This was of course regarded as a huge success of Maxwell's theory.

The fly in the ointment was that (6.1.3) is not Galilean invariant and so if different inertial observers are related by (6.1.2) then only one of them could agree with Maxwell's prediction and the others should measure different speeds, in much the same way that the measured speed of water waves depends very much on the speed of the observer's motion relative to the water.

This led to attempts to use the measured speed of light to infer the Earth's motion relative to whatever the rest frame was of the medium (the 'luminiferous aether') through which these waves move. The absence of any evidence of motion (despite the speed and direction of the Earth's velocity changing as it orbits the Sun) eventually led to the conclusion that perhaps (6.1.2) is not the correct expression relating inertial observers.

Einstein (and Lorentz) found the correct transformation rule relating inertial frames by demanding that all inertial observers must measure precisely the same numerical value, $c = 299,792,458$ m/s, for the speed of light in vacuum. This would allow Maxwell's equations to predict the speed of light correctly for all inertial reference frames (as the measurements seemed to indicate that it did). Einstein's observation was that this could be done if inertial observers disagree on the rate of passage of time.

Because all inertial observers measure the same value for c it becomes possible to define our units of distance so that $c = 1$. Such units would not be useful if all inertial observers did not agree on the speed of light. These units are used throughout the rest of this section. Conversion of subsequent formulae to ordinary units is accomplished by inserting whatever factors of c are required to give the expression the correct dimensions. (*E.g.* for a result like $v = 0.2$ to have the dimensions of m/s, its right-hand-side must really be $0.2\,c$. Similarly, for E an energy and m a mass a formula like $E = m$ becomes $E = m\,c^2$.)

From a symmetry point of view, it is useful to cast the required transformation in a manner that is similar to the way that rotations were treated in §2.3.4: as a linear transformation that mixes up the coordinates but in this case allowing both spatial and temporal components of vectors to mix. Denoting the time and space coordinates by $\{x^\mu\}$ with $\mu = 0, 1, 2, 3$ and $\{x^0, x^1, x^2, x^3\} = \{t, x, y, z\}$ when relating the coordinates of two inertial frames in relative motion we seek a transformation rule of the linear form

$$x^\mu \to \Lambda^\mu{}_\nu \, x^\nu \,, \tag{6.1.4}$$

where the summation convention is in force and the constant matrices $\Lambda^\mu{}_\nu$ are chosen to ensure that all inertial observers agree on the speed of light.

To better formulate the condition that the speed of light remains unchanged it is useful to more precisely specify the physical distances associated with different coordinate displacements Δx^μ. Mathematically this is done by endowing spacetime with a metric — that is, a position-dependent 4×4 matrix, $g_{\mu\nu}(x)$, in terms of which the distance $\mathrm{d}s$ corresponding to a given set of infinitesimal coordinate displacements, $\mathrm{d}x^\mu$, locally is

$$\mathrm{d}s^2 = g_{\mu\nu}(x) \, \mathrm{d}x^\mu \mathrm{d}x^\nu \,. \tag{6.1.5}$$

The summation convention is again in force, with repeated Greek indices (like μ and ν in the above) being summed over the values $0, 1, 2, 3$.

From this point of view special relativity can be summarized as the statement that for *all* inertial observers the metric that appears in (6.1.5) is independent of position and given everywhere by

$$\mathrm{d}s^2 = \eta_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \,, \tag{6.1.6}$$

and so in matrix form for rectangular coordinates $\{x^0, x^1, x^2, x^3\} = \{t, x, y, z\}$, we have

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} , \tag{6.1.7}$$

where all non-written entries are zero. Any 4×4 symmetric invertable matrix $g_{\mu\nu}$ appearing in an expression like (6.1.6) that locally defines the notion of physical distance is called a *spacetime metric* and the particular case $\eta_{\mu\nu}$ is called the *Minkowski* metric.

Notice that there are three cases to explore:

- *Spacelike* separations satisfy $\mathrm{d}s^2 > 0$ and agree with our notion of distance in flat space if it is restricted to a purely spatial interval, along which $\mathrm{d}t = 0$.

- *Lightlike* separations satisfy $\mathrm{d}s^2 = 0$ and describe the trajectory of a light ray. That is, $\mathrm{d}s = 0$ implies $\mathrm{d}t^2 = \mathrm{d}\ell^2$, where $\mathrm{d}\ell^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ measures the spatial distance traversed. Any such a trajectory satisfies $\mathrm{d}\ell/\mathrm{d}t = 1$, and so moves at the speed of light (since $c = 1$). The requirement that all inertial observers agree on the interval $\mathrm{d}s^2$ therefore includes as a special case the condition that all such observers agree on the speed of light.

- *Timelike* separations are those for which $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\ell^2 < 0$. In this case the interval corresponds to the world line of a trajectory of a particle moving at less than the speed of light, since $v^2 = (\mathrm{d}\ell/\mathrm{d}t)^2 = 1 + (\mathrm{d}s/\mathrm{d}t)^2 < 1$. In this case it is useful to define $\mathrm{d}\tau = \sqrt{-\mathrm{d}s^2}$, since this represents the proper time elapsed by the observer moving along this trajectory (for whom $\mathrm{d}\ell = 0$).

The transformations of special relativity are those transformations of the form (6.1.4) that do not change the Minkowski metric eq. (6.1.6). All such observers will agree on physical distances and so also agree on physical laws that are expressed in terms of them. Special Relativity can then be regarded as the requirement that physics looks the same when expressed in terms of either coordinates, x^μ and x'^μ, when the transformation that relates them preserves the Minkowski metric of (6.1.6). Invariance of the metric implies the constant matrices $\Lambda^\mu{}_\nu$ satisfy

$$\eta_{\alpha\beta}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu = \eta_{\mu\nu}\,. \tag{6.1.8}$$

The group of transformations defined by eqs. (6.1.8) are called the *Lorentz group*, or the group $O(3,1)$. If spatial and temporal translations are also included, with $x^\mu \to x^\mu + a^\mu$ for some constant 4-vector a^μ then the symmetry group is called the *Poincaré* group.

Spatial rotations provide a special case of Lorentz transformations, for which

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & \\ & R^i{}_j \end{pmatrix}\,, \tag{6.1.9}$$

where $i, j = 1, 2, 3$ runs over purely spatial directions, and $R^i{}_j$ is an arbitrary 3×3 orthogonal matrix: $\delta_{ij}M^i{}_kM^j{}_l = \delta_{kl}$ (that is to say: R is a rotation). By contrast, the boosts that relate inertial observers with different speeds involves both time and space directions. For instance, as is easy to verify, the matrix

$$(\Lambda_x)^\mu{}_\nu = \begin{pmatrix} \cosh\beta_x & \sinh\beta_x & & \\ \sinh\beta_x & \cosh\beta_x & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\,, \tag{6.1.10}$$

that mixes the t and x directions satisfies the defining condition (6.1.8), for any value of the real parameter β_x. The same is true for the following matrices that similarly mix t with the y and z directions,

$$(\Lambda_y)^\mu{}_\nu = \begin{pmatrix} \cosh\beta_y & & \sinh\beta_y & \\ & 1 & & \\ \sinh\beta_y & & \cosh\beta_y & \\ & & & 1 \end{pmatrix} \quad \text{and} \quad (\Lambda_z)^\mu{}_\nu = \begin{pmatrix} \cosh\beta_z & & & \sinh\beta_z \\ & 1 & & \\ & & 1 & \\ \sinh\beta_z & & & \cosh\beta_z \end{pmatrix}\,, \tag{6.1.11}$$

for any real value of β_y and β_z.

To make contact between the above definitions and the transformations that arise in introductory discussions of special relativity, we must establish how the parameters β_i are related to the components v_i of relative velocity of the two observers. To this end consider the motion of a free particle in the absence of applied forces. Such a particle does not accelerate and so its trajectory in spacetime is given by a straight line,

$$x^\mu(u) = c^\mu + v^\mu \, u \,, \tag{6.1.12}$$

where c^μ and v^μ are constant 4-vectors and u is a parameter that labels the points along the line. Any such a curve satisfies

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}u} = v^\mu \quad \text{and} \quad \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}u^2} = 0 \,. \tag{6.1.13}$$

Geometrically $\mathrm{d}x^\mu/\mathrm{d}u$ gives the components of the tangent vector to this curve, and it is because this is constant that we know eq. (6.1.12) describes a straight world-line.

The invariant interval measured using the metric (6.1.6) along the trajectory is

$$\mathrm{d}s^2 = \eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}u} \frac{\mathrm{d}x^\nu}{\mathrm{d}u} \, \mathrm{d}u^2 = (v \cdot v) \, \mathrm{d}u^2 \,, \tag{6.1.14}$$

so it follows that v^μ must satisfy $v \cdot v = \eta_{\mu\nu} \, v^\mu v^\nu < 0$ for a timelike trajectory (*i.e.* motion with speed less than the speed of light). Such vectors are also said to be timelike. (By contrast, for motion at the speed of light — such as for a photon — v^μ would instead be null: $v \cdot v = 0$.) A curve with a timelike tangent is called a timelike curve.

By definition of the metric the arc-length along any timelike curve defines the proper time, τ, as measured by a clock that moves along this trajectory. Since the invariant interval is negative for time-like trajectories we define $\mathrm{d}\tau^2 = -\mathrm{d}s^2$. It is convenient to use τ rather than u as the parameter labelling points along a timelike curve. In this case $u^\mu := \mathrm{d}x^\mu/\mathrm{d}\tau$ is called the 4-velocity of the trajectory, and eq. (6.1.14) then implies $u \cdot u = -1$.

Writing the components of u^μ as

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = u^\mu = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}x}{\mathrm{d}\tau}, \frac{\mathrm{d}y}{\mathrm{d}\tau}, \frac{\mathrm{d}z}{\mathrm{d}\tau} \right) = \frac{\mathrm{d}t}{\mathrm{d}\tau} \left(1, \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t} \right) \,, \tag{6.1.15}$$

the condition $u \cdot u = -1$ implies $\mathrm{d}t/\mathrm{d}\tau$ satisfies $(\mathrm{d}t/\mathrm{d}\tau)^2(1 - \mathbf{v}^2) = 1$, where the velocity 3-vector, $\mathbf{v}$, is defined to have components $v^i = \mathrm{d}x^i/\mathrm{d}t$. We read off from this the *time dilation* that relates the proper time τ to the time t of the observer with respect to which the trajectory has velocity $\mathbf{v}$:

$$\gamma := \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - \mathbf{v}^2}} \,. \tag{6.1.16}$$

where the condition $\mathrm{d}t/\mathrm{d}\tau > 0$ (*i.e.* both t and τ increase into the future) fixes the sign of the square root.

We may now relate the parameter β appearing in a Lorentz boost to the speed, v, of the inertial observers involved, and thereby verify that eq. (6.1.10) describes a standard Lorentz transformation as derived when still a baby on your mother's knee. To this end, suppose $\Lambda^\mu{}_\nu$ transforms from the frame of an observer at rest (whose 4-velocity is $u^\mu = (1, 0, 0, 0)$) to the frame of an inertial observer moving with speed v along the x axis (whose 4-velocity from (6.1.15) is $u^\mu = (\gamma, \gamma v, 0, 0)$). Then

$$\begin{pmatrix} \gamma \\ \gamma v \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta & & \\ \sinh\beta & \cosh\beta & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{6.1.17}$$

and so $\cosh\beta = \gamma$ and $\sinh\beta = \gamma v$ (and therefore $\tanh\beta = v$). Notice that the definition $\gamma = (1 - v^2)^{-1/2}$ is then equivalent to the identity $\cosh^2\beta - \sinh^2\beta = 1$. β is sometimes called the *rapidity* of the moving particle.

Exercise: Prove the identity $\Lambda_x(\beta_1)\Lambda_x(\beta_2) = \Lambda_x(\beta_1 + \beta_2)$ for the composition of two boosts along the x axis, as in eq. (6.1.10), and use this to show that the inverse of the matrix $\Lambda_x(\beta)$ is $\Lambda_x^{-1}(\beta) = \Lambda_x(-\beta)$. Use your result with the relation $v/c = \tanh\beta$ to derive the relativistic law for adding velocities: if $\beta = \beta_1 + \beta_2$ then

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}. \tag{6.1.18}$$

With this connection between β and v the relation between the coordinates in these two frames, $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$, is

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta & & \\ \sinh\beta & \cosh\beta & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \tag{6.1.19}$$

and so trading β for v (and temporarily replacing the factors of c) gives the familiar expressions

$$t' = \frac{t + vx/c^2}{\sqrt{1 - v^2/c^2}}, \qquad x' = \frac{x + vt}{\sqrt{1 - v^2/c^2}}, \tag{6.1.20}$$

together with $y' = y$ and $z' = z$. Notice that these reduce to Galilean result $t \to t'$ and $x \to x' = x + vt$ – *c.f.* (6.1.2) specialized to motion along the x-axis – in the limit of speeds that are small compared to the speed of light: $v \ll c$.

It is the fact that these expressions imply that events sharing a common value for t are not the same as those sharing a common value for t' – *i.e.* the *relativity of simultaneity* —

that makes it much more efficient to think in terms of spacetime, rather than space and time separately.

Worked example: particle kinematics

The instantaneous 4-momentum, p^μ, of a particle moving slower than the speed of light is proportional to its timelike 4-velocity u^μ,

$$p^\mu = m\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = m\,u^\mu\,, \tag{6.1.21}$$

with proportionality constant $m > 0$. The interpretation of m is found by evaluating the components of p^μ. Using the components for u^μ found in (6.1.15), we have

$$p^0 = E = m\,\gamma = \frac{m}{\sqrt{1-v^2}} \qquad \text{and} \qquad p^i = m\,\gamma\,v^i = \frac{mv^i}{\sqrt{1-v^2}}\,, \tag{6.1.22}$$

which shows that the particle's instantaneous energy is $E = p^0$ and its 3-momentum is p^i and so m is its rest mass. Because (6.1.21) relates 4-vectors to 4-vectors it is true in any inertial frame, which implies the relations (6.1.22) also hold for the components of p^μ and u^μ in any inertial reference frame.

The scalar condition $\eta_{\mu\nu}u^\mu u^\nu = -1$, which holds in all inertial reference frames, implies p^μ defined by (6.1.21) satisfies $\eta_{\mu\nu}p^\mu p^\nu = -m^2$. This is equivalent to the relativistic energy-momentum relation

$$E^2 = \mathbf{p}^2 + m^2\,. \tag{6.1.23}$$

which therefore also holds in all inertial frames.

The 4-momentum of a photon can be thought of as the limit of the above as $m \to 0$ (with $\mathrm{d}\tau \to 0$ so that (6.1.21) doesn't imply $p^\mu \to 0$). The components of p^μ remain fixed and well-defined in this limit, and the $\mathrm{d}\tau \to 0$ limit implies that the 4-velocity $\mathrm{d}x^\mu/\mathrm{d}u$ points in a null direction. Because u^μ is no longer time-like it is no longer possible to choose proper time, τ, as the parameter along the world line. The resulting 4-momentum satisfies $\eta_{\mu\nu}p^\mu p^\nu = p_\mu p^\mu = 0$, and so (6.1.23) reduces to $E = |\mathbf{p}|$.

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Electromagnetism in relativistic notation

Since Lorentz transformations were discovered by requiring consistency with Maxwell's theory of electromagnetism it is no surprise that these can be written in a manifestly Lorentz-covariant way (as this section reviews).

It turns out that the six components of the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, transform as the components of an antisymmetric tensor, $F_{\mu\nu} = -F_{\nu\mu}$, according to

$$\begin{pmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{pmatrix} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}\,, \tag{6.1.24}$$

which labels the inertial coordinates in the usual way, $x^\mu = \{x^0, x^1, x^2, x^3\} = \{t, x, y, z\}$.

There are two types of fundamental laws in electromagnetism. One type expresses the forces felt by charges in the presence of electric and magnetic fields, and states that a point charge of magnitude q moving with velocity $\mathbf{v}$ experiences a *Lorentz force* of magnitude

$$\mathbf{F} = q\Big(\mathbf{E} + \mathbf{v} \times \mathbf{B}\Big)\,. \tag{6.1.25}$$

The second type relates the properties of the electric and magnetic fields to the distribution of charges and currents that source them, as summarized by Maxwell's equations:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,, \qquad \nabla \cdot \mathbf{B} = 0 \tag{6.1.26}$$

and

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}\,, \qquad \nabla \cdot \mathbf{E} = \sigma\,. \tag{6.1.27}$$

Since all inertial observers agree on the laws of electromagnetism, it should be possible to formulate these in terms of Lorentz tensors like $F_{\mu\nu}$. Indeed, the Lorentz force, eq. (6.1.25), can also be grouped into a force 4-vector,

$$F_\mu = q F_{\mu\nu} u^\nu\,, \tag{6.1.28}$$

where u^ν denotes the 4-velocity of the point charge. The relativistic version of Newton's Law, $\dot{\mathbf{p}} = \mathbf{F}$, in the presence of this force then is

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = F^\mu = q F^{\mu\nu} u_\nu\,, \tag{6.1.29}$$

where u^ν denotes the 4-velocity of the point charge, $F_{\mu\nu}$ is defined by (6.1.24) and indices are raised and lowered using the Minkowski metric: $u_\mu = \eta_{\mu\nu} u^\nu$ (with the usual implied summation due to the repeated index ν).

The two source-free Maxwell equation, eqs. (6.1.26), can be similarly written as the combined tensor equation

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0\,, \tag{6.1.30}$$

and the remaining two Maxwell equations with sources, eqs. (6.1.27), become

$$\partial_\nu F^{\mu\nu} = j^\mu\,, \tag{6.1.31}$$

where the 4-vector j^μ is the electric current 4-vector (more about which below). Notice that the antisymmetry $F^{\mu\nu} = -F^{\nu\mu}$ implies $\partial_\mu \partial_\nu F^{\mu\nu}$ vanishes identically, showing that eq. (6.1.31) necessarily implies $\partial_\mu j^\mu = 0$.

Worked example: electromagnetic current conservation

The condition $\partial_\mu j^\mu = 0$ provides the relativistic expression of something familiar: conservation of electric charge.

The components of the electric charge density $\mathfrak{q}(x,t)$ and the electric current $\mathbf{j}(x,t)$ transform together as a 4-vector under Lorentz transformations, with components:

$$j^\mu = \begin{pmatrix} j^0 = \mathfrak{q} \\ j^i \end{pmatrix} , \tag{6.1.32}$$

where j^i represent the 3 spatial components of the current density vector, $\mathbf{j}$.

The quantities $\mathfrak{q}$ and $\mathbf{j}$ should be related by Lorentz transformations because if there is an observer who sees a nonzero density of electric charge, $\mathfrak{q}(x,t)$, then anyone else who moves relative to this observer must see a nonzero electric current density, $\mathbf{j}(x,t)$, in addition to seeing a charge density which differs from the stationary observer (due to the Lorentz contraction of space in the direction of motion).

Being a 4-vector means that it transforms under a Lorentz transformation in a very specific way:

$$j^{\mu'} = \Lambda^\mu{}_\nu \, j^\nu , \tag{6.1.33}$$

and so in the specific case of a boost between inertial observers moving at relative speed $\mathbf{v}$, *c.f.* eqs. (6.1.10) and (6.1.20), this becomes

$$\mathfrak{q}' = j^{0'} = \frac{\mathfrak{q} + \mathbf{v}\cdot\mathbf{j}/c^2}{\sqrt{1-v^2/c^2}} , \qquad \mathbf{j}' = \frac{\mathbf{j} + \mathbf{v}\mathfrak{q}}{\sqrt{1-v^2/c^2}} , \tag{6.1.34}$$

Conservation of electric charge may be expressed in terms of this 4-vector in a manifestly Lorentz-invariant way, as

$$\partial_\mu \, j^\mu = \frac{\partial j^0}{\partial t} + \nabla\cdot\mathbf{j} = 0 . \tag{6.1.35}$$

Since the left-hand side is a Lorentz scalar, if any observer finds the right-hand-side vanishes, then all inertial observers must find that it vanishes. Equation (6.1.35) expresses local charge conservation, as may be seen by integrating it over a volume V having boundary ∂V, and using Gauss' theorem

$$0 = \int_V \left[\frac{\partial j^0}{\partial t} + \nabla\cdot\mathbf{j} \right] \mathrm{d}^3x = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathfrak{q} \, \mathrm{d}^3x + \int_{\partial V} \mathbf{n}\cdot\mathbf{j} \, \mathrm{d}S , \tag{6.1.36}$$

where $\mathrm{d}S$ denotes an infinitesimal area element of the surface, whose outward-pointing normal vector is $\mathbf{n}$. Written this way it is clear that charge is conserved, inasmuch as the rate of change of the total charge in any volume V is equal to the net flux of charge carried by the current through the boundaries of V.

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Finally, the connection between $\mathbf{E}$ and $\mathbf{B}$ and the electromagnetic potentials, Φ and $\mathbf{A}$,

$$\mathbf{B} = \nabla\times\mathbf{A} \qquad \text{and} \qquad \mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} , \tag{6.1.37}$$

can also be grouped into the single tensor equation

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \tag{6.1.38}$$

with the *gauge potential 4-vector* defined by

$$A^\mu = \{A^0, A^i\} = \{\Phi, \mathbf{A}\} \quad \text{and so} \quad A_\mu = \eta_{\mu\nu}A^\nu = \{A_0, A_i\} = \{-\Phi, \mathbf{A}\} . \tag{6.1.39}$$

Exercise: Verify that eqs. (6.1.25), (6.1.26), (6.1.27) and (6.1.37) follow from eqs. (6.1.28), (6.1.30), (6.1.31) and (6.1.38), together with the definitions of $F_{\mu\nu}$, A_μ and j^μ.

6.2 Relativistic Point Particle

We are now in a position to identify the Lagrangian for a point particle from which equations like (6.1.21) emerge as consequences of the Euler-Lagrange equations. We can also do so for electrically charged particles in the presence of electromagnetic fields, and show how equations like (6.1.29) emerge in this way.

6.2.1 Particle Kinematics

We start with a single particle whose spacetime position (or world-line) is given by a 4-vector $x^\mu(u)$, where u is a parameter that labels the different points along the particle's world-line. We seek an action that is invariant under general Poincaré transformations of the form

$$S[x(t)] = \int_C \mathrm{d}u \, L(x, \dot{x}, u) \qquad (6.2.1)$$

where C is the particle world line and the integration is over the parameter u that labels points along the world line. One might think the action should arise as an integral over time, but in relativity there is no unique time slicing of spacetime, so we instead must over a parameterization of the particle world line.

Integrating over u makes sense because parameterizations of a timelike world line can be regarded as clocks that are a proxy for the world line's proper time, $\tau(u)$. Because the parameterization is general we also demand the action be invariant under reparameterizations of the worldline: $u \to v(u)$. Because $\mathrm{d}u \to (\mathrm{d}u/\mathrm{d}v)\,\mathrm{d}v$ under a reparameterization $u \to v(u)$ invariance of S implies the Lagrangian must transform as $L \to (\mathrm{d}v/\mathrm{d}u)L$.

Proceeding along the same lines as was done in §6.1.1 for Galilean transformations we ask how L is restricted by these conditions. Although in principle the Lagrangian L could be a function of x^μ and $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}u$, invariance under spacetime translations, $x^\mu \to x^\mu + a^\mu$ implies that it can only depend on $\dot{x}^\mu$. Lorentz invariance then requires L be unchanged by the replacement $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ and so must have the form

$$L = L(\dot{x}^2) \quad \text{where} \quad \dot{x}^2 := \eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}u} \frac{\mathrm{d}x^\nu}{\mathrm{d}u} = -\left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^2 + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} \,. \qquad (6.2.2)$$

The final symmetry constraint to impose is to ask L to be invariant under reparameterizations of the worldline itself: $u \to v(u)$, since we are free to parameterize it as we like. As mentioned above, this implies $L \to (\mathrm{d}v/\mathrm{d}u)L$ under such a transformation. But $\dot{x}^2 \to (\mathrm{d}v/\mathrm{d}u)^2 \dot{x}^2$ so it follows that L must be proportional to $\sqrt{-\dot{x}^2}$ (with the sign in the square root chosen because $\dot{x}^2 < 0$ for a timelike curve). The resulting action then is

$$S = -m \int_C \mathrm{d}u \, \sqrt{-\eta_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu} = -m \int_C \mathrm{d}u \, \sqrt{-\dot{x}^2} \,, \qquad (6.2.3)$$

where m is a constant of proportionality (that we show below is the particle's rest mass). The overall sign in (6.2.3) is chosen so that it agrees with the sign of (6.1.1) in the nonrelativistic limit.

This argument shows that the form for the action for a single particle is determined up to normalization by the assumption that it only involve first derivatives and that the result be Poincaré invariant (as required for consistency with special relativity). The result was also unique (up to normalization) for Galilean invariance so the difference here arises because the explicit transformation laws relating different inertial frames is different.

The action (6.2.3) has a very simple geometrical interpretation, as is most easily seen by reparameterizing the world-line C using arc-length along the curve. Since the curve is timelike the relevant arc-length is the proper time, τ, for the moving particle, defined by $\mathrm{d}\tau = \sqrt{-\mathrm{d}s^2} = \sqrt{-\dot{x}^2}\,\mathrm{d}u$ where the second equality uses the definition (6.1.6) of the invariant distance, specialized to a curve $x^\mu(u)$ along which $\mathrm{d}x^\mu = \dot{x}^\mu\,\mathrm{d}u$. Comparing this to (6.2.3) shows that the action can alternatively be written

$$S = -m \int_C \mathrm{d}\tau \tag{6.2.4}$$

and so is proportional to the total proper time elapsed along the world line C.

The equations governing the motion of an isolated relativistic particle are given (as usual) by finding the curve $x^\mu(u)$ that extremizes the action (6.2.3). Taking the difference $S[x(u) + \delta x(u)] - S[x(u)]$ and expanding out to linear order in $\delta x^\mu(u)$ then gives

$$\begin{aligned} \delta S[x(u)] &= m \int_{u_0}^{u_f} \mathrm{d}u \left[\frac{\eta_{\mu\nu}\dot{x}^\nu}{\sqrt{-\dot{x}^2}} \right] \delta \dot{x}^\mu \\ &= \left[\frac{m\,\eta_{\mu\nu}\dot{x}^\mu \delta x^\nu}{\sqrt{-\dot{x}^2}} \right]_{u_0}^{u_f} - \eta_{\mu\nu} \int_{u_0}^{u_f} \mathrm{d}u \left[\frac{\mathrm{d}}{\mathrm{d}u} \left(\frac{m\dot{x}^\nu}{\sqrt{-\dot{x}^2}} \right) \delta x^\mu \right] . \end{aligned} \tag{6.2.5}$$

The resulting stationarity conditions are simplest when the parameter u is chosen to be proper time, since in this case $-\dot{x}^2 = 1$, leaving

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = 0 \quad \text{for all } \tau \tag{6.2.6}$$

together with the boundary condition

$$\left[\eta_{\mu\nu} \, \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \, \delta x^\nu \right]_{u_0}^{u_f} = 0\,. \tag{6.2.7}$$

The general solution to (6.2.6) is a straight line in spacetime:

$$x^\mu(\tau) = x_0^\mu + u^\mu\,\tau\,, \tag{6.2.8}$$

where x_0^μ and u^μ are constant 4-vectors and use of the proper time as parameter requires $\dot{x}^2 = -1$ and so implies the 4-velocity u^μ satisfies $\eta_{\mu\nu}\,u^\mu u^\nu = -1$. We see in this way how the action (6.2.3) implies the properties of inertial motion assumed in (6.1.12) and (6.1.13). Just like for the case of Galilean relativity, special relativity again implies Newton's first law: in the action of an applied force an isolated object moves in a straight line with constant speed.

To identify the meaning of the parameter m we differentiate (6.2.3) to get the generalized momentum. This gives

$$p_\mu(u) = \frac{\delta S}{\delta \dot{x}^\mu(u)} = \frac{m}{\sqrt{-\dot{x}^2}}\,\eta_{\mu\nu}\,\dot{x}^\nu\,. \tag{6.2.9}$$

Specializing to proper time as parameter then says $-\dot{x}^2 = 1$ and so $p^\mu = \eta^{\mu\nu} p_\nu = m\,(\mathrm{d}x^\mu/\mathrm{d}\tau) = m\,u^\mu$ (where $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ is the particle's 4-velocity), in agreement with (6.1.21).

Instead specializing to using $x^0 = t$ as the parameter along the world-line instead implies $\mathrm{d}x^0/\mathrm{d}u = \mathrm{d}x^0/\mathrm{d}t = 1$ while $\mathrm{d}x^i/\mathrm{d}u = \mathrm{d}x^i/\mathrm{d}t = v^i$ become the components of the particle's 3-velocity $\mathbf{v}$. In this case $-\dot{x}^2 = 1 - v^2$ where $v^2 = \mathbf{v}\cdot\mathbf{v}$, and so (6.2.9) implies

$$p^\mu = \eta^{\mu\nu} p_\nu = \frac{m}{\sqrt{1-v^2}}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}t} \quad \text{and so} \quad p^0 = \frac{m}{\sqrt{1-v^2}} \quad \text{and} \quad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-v^2}}\,, \tag{6.2.10}$$

in agreement with standard formulae for the velocity dependence of the particle's energy and 3-momentum (compare with (6.1.22)). These expressions confirm the interpretation of the parameter m as the particle's rest mass.

In these same coordinates the action itself becomes $S = -m\int \mathrm{d}t\,\sqrt{1-v^2}$, showing that $S \simeq \frac{1}{2}m\int \mathrm{d}t v^2$ agrees with (6.1.1) (up to an irrelevant additive constant) in the non-relativistic limit $v^2 \ll 1$.

6.2.2 Particle moving in a Gravitational field

The above line of argument generalizes very easily to a particle moving in a gravitational field, though showing this involves using a few facts from general relativity that lie outside our main line of development.

There are three new facts that are pertinent for describing particle motion in a gravitational field. The first states that for relativistic systems the gravitational field is described by a spacetime *metric*, $g_{\mu\nu}(x)$, whose role is to locally define distances in the same way as is done for special relativity in eq. (6.1.6). That is:

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,. \tag{6.2.11}$$

The main difference between this and (6.1.6) is that the value of $g_{\mu\nu}(x)$ can vary from place to place in spacetime.

Now it is also true that even for Special Relativity the entries of the matrix that defines invariant distance can be made to vary from place to place just by doing a coordinate transformation – for example just changing to polar coordinates in (6.1.6) changes it to $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$. But it is *not* true that any given metric $g_{\mu\nu}(x)$ can always be put into the form $g_{\mu\nu} = \eta_{\mu\nu}$ everywhere in spacetime just by doing a coordinate transformation. When this is possible throughout a region then that region is said to have zero spacetime curvature (or to be a *flat* spacetime). Special Relativity emerges from General Relativity as the special case where the metric is flat, and this is the second important fact that we need.

The third important fact is that the equations of motion in General Relativity are the same when written in *arbitrary* coordinate systems. In that sense they remove the need for there to be a proviso that equations of motion only take the same form when written in an inertial reference frame.

These facts allow us to figure out the action governing the motion of a point particle moving in a gravitational field. The dynamical variable again is the timelike curve in spacetime, $x^\mu(u)$, along which the particle might move. In a gravitational field we no longer can demand the action be Poincaré invariant, because those are the symmetries of flat space. In principle the action could depend on both x^μ and $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}u$, but we seek an action where the dependence on undifferentiated x^μ arises only through the position dependence of the metric, $g_{\mu\nu}(x)$. We also assume the action depends on the metric and not its derivatives, as should be true at least for relatively weakly varying gravitational fields.[20]

Although it goes beyond the scope of these notes to prove it, these assumptions require the Lagrangian to be built as a function $L = L(\dot{x}^2)$ of the coordinate invariant combination

$$\dot{x}^2 := g_{\mu\nu}(x)\,\dot{x}^\mu \dot{x}^\nu\,, \tag{6.2.12}$$

that generalizes the definition of $\dot{x}^2$ given in (6.2.2) to a general curved space. Invariance of the action under reparameterizations $u \to v(u)$ again dictates that L must be proportional to $\sqrt{-\dot{x}^2}$, leading to the action

$$S = -m \int_C \mathrm{d}u\,\sqrt{-g_{\mu\nu}[x(u)]\,\frac{\mathrm{d}x^\mu}{\mathrm{d}u}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}u}} = -m \int_C \mathrm{d}u\,\frac{\mathrm{d}\tau}{\mathrm{d}u} = -m \int_C \mathrm{d}\tau\,. \tag{6.2.13}$$

Here m is the same constant of proportionality encountered in flat space and the second-last equality uses (6.2.11) to rewrite the argument inside the square root as the square of the derivative $(\mathrm{d}\tau/\mathrm{d}u)^2$ of the proper distance

$$\mathrm{d}\tau^2 = -\mathrm{d}s^2 = -g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,, \tag{6.2.14}$$

measured along the curve C. Just as in flat spacetime the point particle action is proportional to the proper time along the particle's world line.

The equations of motion for the particle are found as before, by requiring the variation of this action vanish when varying the trajectory $x^\mu(u) \to x^\mu(u) + \delta x^\mu(u)$. Repeating the derivation that in flat space led to (6.2.6) now instead gives

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\lambda}[x(\tau)]\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0 \quad \text{for all } \tau \tag{6.2.15}$$

[20]This assumption precludes interactions that depend on spacetime curvature, for instance.

where the quantity $\Gamma^\mu_{\nu\lambda}(x)$ is given in terms of the metric, $g_{\mu\nu}$, its inverse $g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu$ and its derivatives $\partial_\lambda g_{\mu\nu}$ by

$$\Gamma^\mu_{\nu\lambda} = \tfrac{1}{2}\, g^{\mu\sigma}\Big(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda}\Big)\,. \tag{6.2.16}$$

The attentive reader will recognize some of these expressions as arising in the discussion surrounding eq. (2.1.27), though in a completely different context. Eq. (6.2.15) reduces (as it should) to eq. (6.2.6) in the limit where $g_{\mu\nu} = \eta_{\mu\nu}$ because $\Gamma^\mu_{\nu\lambda}$ vanishes when the metric is constant.

Eq. (6.2.15) has a nice geometrical interpretation: it is the equation for an affinely parameterized geodesic for the metric $g_{\mu\nu}$. As the above derivation shows, these are the curves that extremize the path length between two points as measured with the metric $g_{\mu\nu}$. For spacelike separated curves the arc length along a geodesic provides the minimum distance between these two points (and this is why they turn out to be straight lines for flat spacetimes). For timelike curves the geodesics provide the maximum possible arc length (or proper time) between two fixed endpoints (and this is why accelerated observers always age slower than those that move along geodesics – as in the twin paradox).

Eq. (6.2.15) turns out to have the same form when written in *any* coordinates because the left-hand side of the equation transforms like a rank-one tensor (as it turns out). But the terms $\ddot{x}^\mu$ and $\Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda$ are not separately tensors, so different coordinate systems contribute to each of these differently (but in a way where the non-tensor difference drops out of their sum). Observers using different coordinates calculate different expressions for $\ddot{x}^\mu$ because they differ in the value taken by $\Gamma^\mu_{\nu\lambda}$. For instance, on flat space $\Gamma^\mu_{\nu\lambda}$ vanishes for an inertial observer for whom the metric $\eta_{\mu\nu}$ is constant. But it does not vanish in an accelerating or rotating reference frame. In the language of §3, the fictitious forces are contained within $\Gamma^\mu_{\nu\lambda}$.

6.2.3 Particle interactions with Electromagnetic fields

Since electromagnetism is the poster child of a relativistic theory (from which the form of Lorentz transformations was initially derived), it should be no surprise to find that the interaction term (2.6.16) for a particle interacting with an electromagnetic field needs no changing to be written in a relativistic form, since

$$\begin{aligned} -\int_{t_0}^{t_f} \mathrm{d}t\, V &= q\int_{t_0}^{t_f} \mathrm{d}t\, \Big\{-\Phi[\mathbf{r}(t),t] + \dot{\mathbf{r}}\cdot\mathbf{A}[\mathbf{r}(t),t]\Big\} = q\int_{t_0}^{t_f} \mathrm{d}t\, \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\, A_\mu[x_a(t)] \\ &= q\int_{u_0}^{u_f} \mathrm{d}u\, \frac{\mathrm{d}x^\mu}{\mathrm{d}u}\, A_\mu[x_a(u)]\,, \end{aligned} \tag{6.2.17}$$

since $x^\mu = \{t\,,\mathbf{r}\}$ and $A_0 = -\Phi$ (*c.f.* eq. (6.1.39)). In the top line the time coordinate t is used as a parameter to label points along the particle trajectory, since this is the standard practice for nonrelativistic mechanics. The second line uses the fact that the integral is parameter invariant – *i.e.* the Jacobian $\mathrm{d}t/\mathrm{d}u$ cancels between the integration measure and the derivative

– to relabel the curve using an arbitrary parameter u (which can, but need not, be chosen to be arc-length – or proper time – as measured along the curve itself).

The only thing needed to make particle motion in an electromagnetic field manifestly relativistic in this case is to use the relativistic form (6.2.3) or (6.2.4) for the particle kinetic energy, rather than (say) (6.1.1). We are led in this way to the relativistic generalization of the point-particle Lagrangian (2.6.22) in the presence of an electromagnetic field:

$$S = \int_C \mathrm{d}u \Big\{ -m\sqrt{-\eta_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu} + q\,\dot{x}^\mu A_\mu[x(t)] \Big\}\,, \tag{6.2.18}$$

where (as usual) C denotes the world-line of the particle in question.

The interpretation of the parameter q can be found by examining the equations of motion predicted by (6.2.18), implied by requring $\delta S = 0$ for arbitrary variations $\delta x^\mu(u)$ of the particle world line. These are (compare with (6.2.5))

$$\eta_{\mu\nu}\left[\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{m\dot{x}^\nu}{\sqrt{-\dot{x}^2}}\right)\right] + q\frac{\mathrm{d}x^\nu}{\mathrm{d}u}(\partial_\nu A_\mu - \partial_\mu A_\nu) = 0\,, \tag{6.2.19}$$

which when specialized to proper-time parameterization ($u = \tau$ for which $\dot{x}^2 = \eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$) gives

$$m\,\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} = qF^\mu{}_\nu\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\,. \tag{6.2.20}$$

Here $F^\mu{}_\nu = \eta^{\mu\lambda}F_{\lambda\nu}$. This agrees with (6.1.29) once we use the definition (6.1.21) of 4-momentum, which tells us that q is the particle's electric charge.

6.3 Strings and Membranes*

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7 Hamiltonian Mechanics

The Hamiltonian formulation of mechanics starts from the Lagrangian formulation, but rather than treating the generalized velocities, $\dot{q}^A$, as secondary to the 'positions' q^A the idea is to treat them as being on more of an equal footing with the generalized positions. Even better is to focus directly on the more useful, potentially conserved, quantities like generalized momenta p_A, rather than using velocities.

Part of the motivation for entertaining this type of reformulation is to allow an even broader class of variable changes when analyzing physical situations. After all, the Lagrangian formulation's superpower is the ease with which it handles changes of variables, and this ability to find efficiently the most convenient variables to simplify a problem can be the difference between being able to solve it or not. But to this point the main freedom to change variables has taken the form of redefinitions of the generalized coordinates $q^A \to \tilde{q}^A = f^A(q)$, with the velocities then just going along for the ride by differentiation: $\dot{\tilde{q}}^A = (\partial f^A/\partial q^B)\,\dot{q}^B$.

Putting q^A and p_B on a more equal footing opens up the possibility for handling an even broader class of redefinitions that mix up both positions and momenta: $\{p,q\} \to \{\tilde{p}(p,q), \tilde{q}(p,q)\}$. We shall see that doing so also allows us to rewrite time evolution purely in terms of first-order differential equations rather than second-order ones, though at the expense of having more variables to integrate.

7.1 Canonical Evolution

To see how this works, recall the definition (2.3.2) of the generalized momentum p_A for each q^A, and the Euler-Lagrange equation (2.2.4):

$$p_A := \frac{\partial L}{\partial \dot{q}^A} \quad \text{in terms of which the Euler-Lagrange equation is} \quad \dot{p}_A = \frac{\partial L}{\partial q^A}\,. \tag{7.1.1}$$

As usual we here imagine that $L(q,\dot{q},t)$ is a known function and the partial derivative with respect to $\dot{q}$ denotes differentiation with the other variables – in this case q and t – held fixed. What we seek is a reformulation of the least-action principle for which it is $q^A(t)$ and $p_A(t)$ that are independently varied, rather than q^A and $\dot{q}^A$. That is, we seek the a quantity $H(q,p,t)$ – called the system's *Hamiltonian* – whose natural arguments[21] are $q^A(t)$ and $p_A(t)$, in the sense that derivatives with respect to q^A are taken with p_B fixed rather than fixing $\dot{q}^B$.

To motivate the construction recall that the variation of the Lagrangian under a change of path can be written

$$\delta L(q,\dot{q},t) = \left(\frac{\partial L}{\partial \dot{q}^A}\right)\delta\dot{q}^A + \left(\frac{\partial L}{\partial q^A}\right)\delta q^A = p_A\,\delta\dot{q}^A + \dot{p}_A\,\delta q^A\,, \tag{7.1.2}$$

where (as always) do not forget the implied sum over the repeated index A and the second equality uses (7.1.1). We'd like to trade the $\delta\dot{q}^A$ term for a δp_A term, so recalling that the variation of $p_A\dot{q}^A$ is $\delta(p_A\dot{q}^A) = p_A\,\delta\dot{q}^A + \dot{q}^A\,\delta p_A$ motivates the following definition:

$$H(q,p,t) := p_A\,\dot{q}^A - L(q,\dot{q},t)\,, \tag{7.1.3}$$

for which

$$\delta H = (p_A\,\delta\dot{q}^A + \dot{q}^A\delta p_A) - (p_A\delta\dot{q}^A + \dot{p}_A\delta q^A) = \dot{q}^A\delta p_A - \dot{p}_A\delta q^A\,. \tag{7.1.4}$$

[21] If you have taken thermodynamics you might recognize this question. A similar question gets asked in thermodynamics when defining thermodynamic potentials. A quantity like the Helmholtz free energy, $A(T,V)$, naturally has temperature and volume as arguments, but it is sometimes more useful to express things in terms of temperature and pressure (such as when describing the mechanical equilibrium between two phases of a fluid). In this case it is the Gibbs free energy $G(T,p)$ that is of more interest, and the different descriptions are famously related by a *Legendre transformation*. The same solution applies here: we must perform a Legendre transformation on $L(q,\dot{q},t)$ to obtain the desired quantity $H(q,p,t)$.

Notice that (7.1.3) shows that H agrees with the energy E associated with the lagrangian L, that was defined in (2.3.8) when discussing Noether's theorem and conservation laws.

Whenever $\dot{q}^A$ arises on the right-hand side of (7.1.3) we are supposed to use $\dot{q}^A = \dot{q}^A(q,p)$ as obtained in principle by inverting the definition $p_A = p_A(q,\dot{q})$ obtained from (7.1.1). In the simplest version of the Hamiltonian formalism eqs. (7.1.1) are invertible and so $\dot{q}^A(q,p)$ can always be found in this way. But it can happen that this is sometimes not possible, such as when the system is subject to constraints. (For instance, how does one invert things if a constraint sets $p_A = 0$?). And the possibility that (7.1.1) cannot be inverted is not just an obscure academic footnote since it actually happens in many of the theories we believe actually describe nature at a fundamental level – as we see in §6 and §10.2. In such cases a generalization is necessary for the procedure described here, as described in §11.

Returning to the main argument, to find how Newton's laws are reformulated we write the action $S[p(t), q(t)]$ in terms of $H(p,q,t)$:

$$S = \int_{t_0}^{t_f} \mathrm{d}\tau \; L(\dot{q}, q, \tau) = \int_{t_0}^{t_f} \mathrm{d}\tau \; \Big[p_A \, \dot{q}^A - H(q,p,\tau) \Big] \,. \tag{7.1.5}$$

As usual, this is regarded as a functional that assigns a number to any given particle trajectory but with the important new proviso that the trajectory is now specified by giving both $q^A(t)$ and $p_B(t)$ (and not assuming one can be obtained from the other by differentiating) rather than just giving $q^A(t)$ alone.

Varying the action and demanding that it be stationary with respect to arbitrary small variations $\delta q^A(\tau)$ and $\delta p_B(\tau)$ gives the desired reformulation of Newton's laws. Explicitly

$$\begin{aligned} \delta S &= \int_{t_0}^{t_f} \mathrm{d}\tau \left[\delta p_A \, \dot{q}^A + p_A \, \delta \dot{q}^A - \frac{\partial H}{\partial q^A} \, \delta q^A - \frac{\partial H}{\partial p_A} \, \delta p_A \right] \\ &= \Big[p_A \delta q^A \Big]_{t_0}^{t_f} + \int_{t_0}^{t_f} \mathrm{d}\tau \left[\left(\dot{q}^A - \frac{\partial H}{\partial p_A} \right) \delta p_A - \left(\dot{p}_A + \frac{\partial H}{\partial q^A} \right) \delta q^A \right] \,. \end{aligned} \tag{7.1.6}$$

Requiring this to vanish for any $\delta q^A(\tau)$ and $\delta p_A(\tau)$ subject to the boundary condition $\delta q^A(t_0) = \delta q^A(t_f) = 0$ then implies Hamilton's first-order reformulation of Newton's equations of motion

$$\dot{q}^A = \frac{\partial H}{\partial p_A} \quad \text{and} \quad \dot{p}_A = -\frac{\partial H}{\partial q^A} \,, \tag{7.1.7}$$

where the partial derivative with respect to q^A now is done with p_B fixed (rather than with the $\dot{q}^A$'s being fixed).[22]

Eqs. (7.1.7) are the Hamiltonian equations of motion, which are remarkably symmetrical in how the q's and p's appear. Because of this symmetry they are sometimes also called the

[22]If the problem also requires the action to be stationary against variations with $\delta q^A(t_0)$ and $\delta q^A(t_f)$ nonzero then the 'surface term' of eq. (7.1.6) shows we must require (7.1.7) to be true *and* impose the additional conditions $p_A(t_0) = p_A(t_f) = 0$.

canonical equations of motion, with the momenta appearing in them – defined by (7.1.1) – called the *canonical momenta* for the generalized coordinates q^A.

In order to see how these definitions reproduce earlier formulations of Newton's laws it is worth reconsidering some simple examples in this new context.

7.1.1 Examples

It is worth seeing in detail how this works in a few familiar examples.

Simple harmonic oscillator

For instance for the 1D harmonic oscillator we had

$$L = \tfrac{1}{2}\, m\dot{x}^2 - \tfrac{1}{2}\, kx^2 \tag{7.1.8}$$

for which the equation of motion obtained from the Euler-Lagrange equation (2.2.4) is

$$m\ddot{x} + kx = 0\,. \tag{7.1.9}$$

For the Lagrangian (7.1.8) the canonical momentum obtained from (7.1.1) is $p = m\dot{x}$, which inverts to give $\dot{x} = p/m$, so the Hamiltonian becomes

$$H(x,p) = p\,\dot{x} - L = \frac{p^2}{m} - \left(\frac{p^2}{2m} - \frac{kx^2}{2}\right) = \frac{p^2}{2m} + \frac{kx^2}{2}\,. \tag{7.1.10}$$

This is to be compared with the equations (7.1.7) obtained from (7.1.10):

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial x} = -kx\,, \tag{7.1.11}$$

of which the first agrees with the above definition, $p = m\dot{x}$, and the second (after eliminating p) again gives $m\ddot{x} + kx = 0$, in agreement with (7.1.9).

Two-body central force

The lagrangian describing the relative motion for two bodies moving under the influence of a conservative central force is (in polar coordinates)

$$L = \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\mu r^2\dot{\theta}^2 + \tfrac{1}{2}\mu r^2\sin^2\theta\,\dot{\phi}^2 - V(r)\,, \tag{7.1.12}$$

for which the canonical momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = \mu\dot{r}\,, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta} \quad \text{and} \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \mu r^2\sin^2\theta\,\dot{\phi}\,, \tag{7.1.13}$$

and the Euler-Lagrange equations become

$$\dot{p}_r = \mu r(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) - V'(r)\,, \quad \dot{p}_\theta = \mu r^2\sin\theta\,\dot{\phi}^2 \quad \text{and} \quad \dot{p}_\phi = 0\,. \tag{7.1.14}$$

Eqs. (7.1.13) invert to give the generalized velocities

$$\dot r = \frac{p_r}{\mu}\,, \quad \dot\theta = \frac{p_\theta}{\mu r^2} \quad \text{and} \quad \dot\phi = \frac{p_\phi}{\mu r^2 \sin^2\theta}\,, \tag{7.1.15}$$

and so the Hamiltonian that results then is

$$H = p_r\,\dot r + p_\theta\,\dot\theta + p_\phi\,\dot\phi - L = \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} + \frac{p_\phi^2}{2\mu r^2\sin^2\theta} + V(r)\,. \tag{7.1.16}$$

The equations of motion that equations (7.1.7) imply in this case are:

$$\dot r = \frac{\partial H}{\partial p_r} = \frac{p_r}{\mu}\,, \quad \dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{\mu r^2} \quad \text{and} \quad \dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{\mu r^2\sin^2\theta}\,, \tag{7.1.17}$$

in agreement with (7.1.15), and

$$\dot p_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{\mu r^3} + \frac{p_\phi^2}{\mu r^3 \sin^2\theta} - V'(r)\,, \quad \dot p_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2}{\mu r^2 \sin^3\theta\cos\theta} \tag{7.1.18}$$

and $\dot p_\phi = -\partial H/\partial\phi = 0$, in agreement with (7.1.14) once (7.1.13) are used.

N-body conservative interactions

As our third example consider the case of N particles interacting through a conservative force that is derivable from a potential energy that is a function only of the particle positions, $V(\mathbf{r}_1, \cdots, \mathbf{r}_N)$.

The lagrangian describing the motion for these particles can be written

$$L = \tfrac12 \sum_a m_a \dot{\mathbf{r}}_a^2 - V(\mathbf{r}_1, \cdots, \mathbf{r}_N)\,. \tag{7.1.19}$$

The canonical momenta obtained from this are

$$\mathbf{p}_a = \frac{\partial L}{\partial \dot{\mathbf{r}}_a} = m_a\,\dot{\mathbf{r}}_a\,, \tag{7.1.20}$$

and the corresponding Euler-Lagrange equations become

$$\dot{\mathbf{p}}_a = -\nabla_a V = -\frac{\partial V}{\partial \mathbf{r}_a}\,. \tag{7.1.21}$$

Eqs. (7.1.20) invert to give $\dot{\mathbf{r}}_a = \mathbf{p}_a/m_a$ and so the Hamiltonian is

$$H = \sum_a \mathbf{p}_a\cdot\dot{\mathbf{r}}_a - L = \sum_a \frac{\mathbf{p}_a^2}{2m_a} + V(\mathbf{r}_1, \cdots, \mathbf{r}_N)\,, \tag{7.1.22}$$

in agreement with the energy for such a system. The Hamiltonian equations of motion (7.1.7) imply in this case:

$$\dot{\mathbf{r}}_a = \frac{\partial H}{\partial \mathbf{p}_a} = \frac{\mathbf{p}_a}{m_a}\,, \tag{7.1.23}$$

in agreement with (7.1.20) above, and

$$\dot{\mathbf{p}}_a = -\frac{\partial H}{\partial \mathbf{r}_a} = -\frac{\partial V}{\partial \mathbf{r}_a} \tag{7.1.24}$$

in agreement with (7.1.21).

7.1.2 The Routhian and Ignorable Coordinates

An immediate consequence of the canonical equations (7.1.7) is that $p_A(\tau)$ is conserved for any q^A that does not appear in $H(q,p)$. An explicit instance when this occurs is provided by Hamiltonian for two-body central-force motion, eq. (7.1.16), which does not depend on the variable ϕ. These are what were called 'ignorable' (or cyclic) coordinates when encoutered in §2.3.1, where they arose whenever L depends on a variable q^A only through its derivative $\dot{q}^A$. Ignorable coordinates are ignorable in the sense that they influence the motion of other coordinates only through the constant value taken by their momentum.

For example, the central force two-body problem ignorable coordinates arise because of angular momentum conservation. This ensures ϕ is an ignorable coordinate with conserved canonical momentum, p_ϕ. In practice this means $\dot{\phi}$ can be eliminated in terms of $p_\phi = J$ using (7.1.15), leading in this case to

$$\dot{\phi} = \frac{J}{\mu r^2}\,. \tag{7.1.25}$$

Using this in the equations of motion for the other variables means they depend on ϕ only through the constant value J. As argued in §1.2.1 angular momentum conservation also forces the orbit to lie in a plane, allowing us to adapt our coordinates so that the orbital plane satisfies $\theta = \frac{\pi}{2}$ for all time (so $p_\theta = 0$). Once this is done the equations for the remaining variable r – *c.f.* eq. (7.1.18) – become

$$\dot{p}_r = \mu\ddot{r} = \frac{J^2}{\mu r^3} - V'(r) = -V'_{\rm eff}(r) \quad \text{where} \quad V_{\rm eff} = \frac{J^2}{2\mu r^2} + V(r)\,, \tag{7.1.26}$$

which we saw completely fixes the radial motion.

There is a subtlety in the removal of ignorable coordinates that this example illustrates. Notice that if we had made the replacement $p_\phi = J$ (and $\theta = \frac{\pi}{2}$) directly in expression (7.1.12) for L it would have given

$$L = \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\mu r^2\,\dot{\phi}^2 - V(r) = \tfrac{1}{2}\mu\dot{r}^2 + \frac{J^2}{2\mu r^2} - V(r)\,, \tag{7.1.27}$$

and this, if varied with respect to r, does *not* reproduce the correct equation (7.1.26) because the J^2 term has the wrong sign. It is crucial only to eliminate the ignorable coordinate *after* all of the other variables have been varied to obtain their equations of motion.

This same subtlety does not occur if the problem is studied in the Hamiltonian formulation, however, because replacing $p_\phi = J$ (and $\theta = \frac{\pi}{2}$) in the Hamiltonian (7.1.16) gives

$$H = \frac{p_r^2}{2\mu} + \frac{J^2}{2\mu r^2} + V(r) = \frac{p_r^2}{2\mu} + V_{\rm eff}(r)\,, \tag{7.1.28}$$

with the J^2 term appearing with the correct sign.

This distinction matters because it is sometimes useful to have a reduced version of the Lagrangian within which ignorable coordinates are already eliminated. A trick for being able to do so is to perform the Legendre transformation from L to H, but only for the ignorable coordinates like ϕ and not for the other coordinates like r. The hybrid quantity obtained from the Lagrangian when only some of the coordinates are transformed in this way is called a *Routhian* rather than a Lagrangian or Hamiltonian. One of its virtues is that ignorable coordinates can be eliminated directly in the Routhian before varying the other variables, leaving a result that can be treated as a Lagrangian for those other degrees of freedom.

Suppose, then, that the generalized coordinates q^A are divided into two groups: $\{q^A\} = \{q^a, \xi^\alpha\}$ and where we perform the transformation from $L(q,\xi,\dot{q},\dot{\xi},t)$ to $R(q,\xi,p,\dot{\xi},t)$ by defining

$$p_a := \frac{\partial L}{\partial \dot{q}^a} \quad \text{and} \quad R(q,\xi,p,\dot{\xi},t) = p_a\dot{q}^a - L(q,\xi,\dot{q},\dot{\xi},t)\,, \tag{7.1.29}$$

where as usual we imagine solving for $\dot{q}^a = \dot{q}^a(q,\xi,p,\dot{\xi},t)$. That is, we perform a Legendre transformation only on the q^a variables (which we imagine might be ignorable, though we do not explicitly use that they are ignorable in the definition (7.1.29)).

Under small changes of the variables this definition of R implies

$$\begin{aligned} \delta R &= (p_a\delta\dot{q}^a + \dot{q}^a\delta p_a) - \left(\frac{\partial L}{\partial \dot{q}^a}\,\delta\dot{q}^a + \frac{\partial L}{\partial q^a}\,\delta q^a + \frac{\partial L}{\partial \dot{\xi}^\alpha}\,\delta\dot{\xi}^\alpha + \frac{\partial L}{\partial \xi^\alpha}\,\delta\xi^\alpha\right) \\ &= \dot{q}^a\delta p_a - \dot{p}_a\delta q^a - \left(\frac{\partial L}{\partial \dot{\xi}^\alpha}\,\delta\dot{\xi}^\alpha + \frac{\partial L}{\partial \xi^\alpha}\,\delta\xi^\alpha\right) \end{aligned} \tag{7.1.30}$$

which implies

$$\dot{q}^a = \left(\frac{\partial R}{\partial p_a}\right)_{q,\xi,\dot{\xi}} \quad \text{and} \quad \dot{p}_a = -\left(\frac{\partial R}{\partial q^a}\right)_{p,\xi,\dot{\xi}} \tag{7.1.31}$$

where the subscripts make explicit what is held fixed when taking the partial derivative. We also learn that derivatives of L and R are related by

$$\left(\frac{\partial R}{\partial \xi^\alpha}\right)_{q,p,\dot{\xi}} = -\left(\frac{\partial L}{\partial \xi^\alpha}\right)_{q,\dot{q},\xi,\dot{\xi}} \quad \text{and} \quad \left(\frac{\partial R}{\partial \dot{\xi}^\alpha}\right)_{q,p,\dot{\xi}} = -\left(\frac{\partial L}{\partial \dot{\xi}^\alpha}\right)_{q,\dot{q},\xi,\dot{\xi}}\,. \tag{7.1.32}$$

Using these in the Euler-Lagrange equations (2.2.4) for ξ^α then implies these can also be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial R}{\partial \dot{\xi}^\alpha}\right) = \frac{\partial R}{\partial \xi^\alpha}\,. \tag{7.1.33}$$

As is easy to verify, equations (7.1.31) and (7.1.33) ensure that the action

$$S = \int_{t_0}^{t_f} \mathrm{d}\tau\,\Big(p_a\dot{q}^a - R\Big) \tag{7.1.34}$$

satisfies $\delta S = 0$ for arbitrary variations δq^a, δp_b and $\delta\xi^\alpha$, showing that they are equations of motion. Evidently $R(q,\xi,p,\dot{\xi},t)$ behaves like a Lagrangian for the variables ξ^α and $\dot{\xi}^\alpha$ but like a Hamiltonian for the variables q^a and p_a.

The energy for this system is found by applying eq. (2.3.8) to the original Lagrangian and using (7.1.32) in the result, leading to

$$E = \left(\frac{\partial L}{\partial \dot\xi^\alpha}\right) \dot\xi^\alpha + \left(\frac{\partial L}{\partial \dot q^a}\right) \dot q^a - L = -\left(\frac{\partial R}{\partial \dot\xi^\alpha}\right) \dot\xi^\alpha + R\,. \tag{7.1.35}$$

As a concrete example, the two-body central force case with $\theta = \frac{\pi}{2}$ has the Lagrangian,

$$L = \tfrac12 \mu \dot r^2 + \tfrac12 \mu r^2\, \dot\phi^2 - V(r)\,, \tag{7.1.36}$$

and performing the Legendre transformation (7.1.29) only for the variable ϕ and not to r leads to the Routhian

$$R(r,\phi,\dot r, p_\phi) = p_\phi \dot\phi - L = -\frac{\mu \dot r^2}{2} + \frac{p_\phi^2}{2\mu r^2} + V(r)\,. \tag{7.1.37}$$

The energy (7.1.35) for this Routhian is

$$E = -\left(\frac{\partial R}{\partial \dot r}\right) \dot r + R = \frac{\mu \dot r^2}{2} + \frac{p_\phi^2}{2\mu r^2} + V(r) = \frac{\mu \dot r^2}{2} + V_{\rm eff}(r)\,, \tag{7.1.38}$$

in agreement with the standard result. The virtue of using R rather than L is that we can set $p_\phi = J$ in R before varying $r(t)$ and still obtain the correct equations governing how r evolves once the ignorable coordinate is removed by replacing p_ϕ with J.

Worked example: Symmetric top in a gravitational field (again)

Consider next the motion of a symmetric top of mass M in a constant gravitational field, supported on its axis a distance ℓ from its centre of mass. In particular we wish to compute the Routhian that describes how the remaining degrees move once these ignorable coordinates have been removed. The Lagrangian for this system expressed in terms of Euler angles is given in (4.5.28), repeated again here

$$L = \tfrac12 \Big[\mathcal{I}_1(\dot\theta^2 + \dot\phi^2 \,\sin^2\theta) + \mathcal{I}_3(\dot\psi + \dot\phi\,\cos\theta)^2\Big] - Mg\ell\,\cos\theta\,. \tag{7.1.39}$$

The ignorable coordinates here are ϕ and ψ, whose momenta

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = \mathcal{I}_3(\dot\psi + \dot\phi\cos\theta) = \mathcal{I}_1 a\,, \tag{7.1.40}$$

and

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = (\mathcal{I}_1\,\sin^2\theta + \mathcal{I}_3\,\cos^2\theta)\dot\phi + \mathcal{I}_3\,\dot\psi\,\cos\theta = \mathcal{I}_1 b\,, \tag{7.1.41}$$

are conserved (with constant values parameterized using the integration constants a and b). Solving these for $\dot\psi$ and $\dot\phi$ then gives

$$\dot\phi = \frac{b - a\cos\theta}{\sin^2\theta} \quad \text{and} \quad \dot\psi = \frac{\mathcal{I}_1 a}{\mathcal{I}_3} - \left(\frac{b - a\cos\theta}{\sin^2\theta}\right)\cos\theta\,. \tag{7.1.42}$$

The Routhian for the variable θ is found by performing the Legendre transformation on the ignorable coordinates ψ and ϕ:

$$\begin{aligned}
R(\theta,\dot\theta,a,b) &= p_\phi\,\dot\phi + p_\psi\,\dot\psi - L \\
&= \mathcal{I}_1 b\left(\frac{b-a\cos\theta}{\sin^2\theta}\right) + \mathcal{I}_1 a\left[\frac{\mathcal{I}_1 a}{\mathcal{I}_3} - \left(\frac{b-a\cos\theta}{\sin^2\theta}\right)\cos\theta\right] \\
&\qquad - \left[\tfrac12\mathcal{I}_1\dot\theta^2 + \tfrac12\mathcal{I}_1\sin^2\theta\left(\frac{b-a\cos\theta}{\sin^2\theta}\right)^2 + \tfrac12\mathcal{I}_3\left(\frac{\mathcal{I}_1 a}{\mathcal{I}_3}\right)^2 - Mg\ell\cos\theta\right] \\
&= -\tfrac12\mathcal{I}_1\dot\theta^2 + Mg\ell\cos\theta + \frac{(\mathcal{I}_1 a)^2}{2\mathcal{I}_3} + \tfrac12\mathcal{I}_1 b\,\frac{(b-a\cos\theta)^2}{\sin^2\theta}\,,
\end{aligned} \tag{7.1.43}$$

The conserved energy for this Routhian is given by (7.1.35), which in this case becomes

$$E = -\dot\theta\left(\frac{\partial R}{\partial\dot\theta}\right) + R = \tfrac12\mathcal{I}_1\dot\theta^2 + Mg\ell\cos\theta + \frac{(\mathcal{I}_1 a)^2}{2\mathcal{I}_3} + \tfrac12\mathcal{I}_1 b\,\frac{(b-a\cos\theta)^2}{\sin^2\theta}\,, \tag{7.1.44}$$

in agreement with (4.5.36), reproduced again here:

$$\dot\theta^2\sin^2\theta = (\alpha-\beta\cos\theta)\sin^2\theta - (b-a\cos\theta)^2\,, \tag{7.1.45}$$

once we write $\mathcal{E} = E - (\mathcal{I}_1 a)^2/(2\mathcal{I}_3)$ and identify $\alpha = 2\mathcal{E}/\mathcal{I}_1$ and $\beta = 2Mg\ell/\mathcal{I}_1$ as in (4.5.37).

⋆ ⋆ ⋆

7.1.3 Higher derivatives and the Ostrogradsky instability

As a different application of canonical methods consider next how to handle variational problems where the Lagrangian is allowed to depend on more than just q and $\dot q$, such as by being allowed to depend also on $\ddot q$. This example sets up the Hamiltonian formalism for such systems and shows them generically to produce a Hamiltonian that is not bounded from below.

When the Hamiltonian is unbounded from below the theory is usually unstable: because there is no minimum energy configuration. Usually we demand theories to have energies that are bounded from below since this is usually a prerequisite for the system to have a ground state to which it can relax. If the energy is not bounded from below then the system can always spontaneously dissipate energy into radiation, and can do so without end because there is no bottom: there is not limit to how negative the energy can go. The observation that higher-derivative theories are prone to this type of instability is known as the *Ostrogradsky* instability, and it is a large part of the reason why we restrict our attention to Lagrangians that depend only on q and $\dot q$ when modeling physical systems.

In order to make this point it suffices to consider only a single variable q. Consider therefore

$$L = L(q,\dot q,\ddot q)\,, \tag{7.1.46}$$

for which the variation of the action $\delta S = \int \mathrm{d}\tau \, \delta L$ becomes

$$\begin{aligned} \delta S &= \int_{t_0}^{t_f} \mathrm{d}\tau \, \left[\frac{\partial L}{\partial q} \, \delta q + \frac{\partial L}{\partial \dot{q}} \, \delta \dot{q} + \frac{\partial L}{\partial \ddot{q}} \, \delta \ddot{q} \right] \\ &= \left\{ \left[\frac{\partial L}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial L}{\partial \ddot{q}} \right) \right] \delta q + \frac{\partial L}{\partial \ddot{q}} \delta \dot{q} \right\}_{t_0}^{t_f} \\ &\qquad + \int_{t_0}^{t_f} \mathrm{d}\tau \, \left\{ \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial L}{\partial \dot{q}} \right) + \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} \left(\frac{\partial L}{\partial \ddot{q}} \right) \right\} \delta q \, . \end{aligned} \tag{7.1.47}$$

Consider first variations $\delta q(\tau)$ that vanish at the endpoints – $\delta q(t_0) = \delta q(t_f) = 0$ – and whose time derivative also vanishes at the endpoints – $\delta \dot{q}(t_0) = \delta \dot{q}(t_f) = 0$. For the action to be stationary with respect to arbitrary variations like these requires the following Euler-Lagrange equation to hold:

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial L}{\partial \dot{q}} \right) + \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} \left(\frac{\partial L}{\partial \ddot{q}} \right) = 0 \, . \tag{7.1.48}$$

If the variation is not required to vanish at the endpoints then (7.1.48) still applies but we also learn from (7.1.47) the additional information that

$$\frac{\partial L}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial L}{\partial \ddot{q}} \right) \quad \text{and} \quad \frac{\partial L}{\partial \ddot{q}} \quad \text{must vanish at } t = t_0 \text{ and } t = t_f. \tag{7.1.49}$$

For second-order equations of motion these would provide too many boundary conditions relative to the number of integration constants that can be used to satisfy them. But because (7.1.48) can involve up to four derivatives of q with respect to time, its integration can involve four integration constants rather than the usual two. It is these additional integration constants that can allow the freedom to ask for additional boundary conditions at the endpoints.

It is worth making the discussion concrete by having a simple example in mind, so consider the following quadratic Lagrangian

$$L = \tfrac{1}{2} \left(-m^2 q^2 + \dot{q}^2 + \frac{\ddot{q}^2}{M^2} \right) \, , \tag{7.1.50}$$

where the freedom to rescale q has been used to set the coefficient of $\dot{q}^2$ to unity. The Euler-Lagrange equation (7.1.48) for this lagrangian is the following fourth-order linear ordinary differential equation:

$$-m^2 q - \ddot{q} + \frac{\eta}{M^2} \ddddot{q} = 0 \, , \tag{7.1.51}$$

where m and M are real parameters with the dimensions of frequency and $\eta = \pm$ is a sign to be chosen to explore the consequences of changing the sign of the 4-derivative term.

This has as its general solution

$$q = \sum_{i=1}^{4} A_i \, e^{-i\omega_i t} \, , \tag{7.1.52}$$

where the integration constants A_i are chosen to reproduce the initial conditions $q(t_0)$, $\dot{q}(t_0)$, $\ddot{q}(t_0)$ and $\dddot{q}(t_0)$ and the frequencies ω_i are the roots to the quartic equation

$$\frac{\eta\omega^4}{M^2} + \omega^2 - m^2 = 0\,. \tag{7.1.53}$$

These are explicitly given by

$$\omega_\pm^2 = \tfrac{1}{2}\eta M^2 \left[-1 \pm \sqrt{1 + \frac{4\eta m^2}{M^2}} \right] , \tag{7.1.54}$$

which for $m \ll M$ become

$$\omega_+^2 \simeq m^2 \left[1 + \mathcal{O}\left(\frac{m^2}{M^2}\right) \right] \quad \text{and} \quad \omega_-^2 \simeq -\eta M^2 \left[1 + \mathcal{O}\left(\frac{m^2}{M^2}\right) \right] . \tag{7.1.55}$$

Consider first $\eta = +1$. Then $\omega_\pm^2$ is always real, and the roots ω_+ are also real and reproduce the standard oscillatory solutions expected in the limit $M \to \infty$ (for which the higher-derivative term in (7.1.50) becomes negligible). In this case the roots ω_- are always imaginary, however, and this points to an instability problem since one of the solutions $e^{-i\omega_- t}$ necessarily grows exponentially in time. Worse, this growth is very rapid as $M \to \infty$, despite large M naively being the limit where the higher-derivative term should become negligible in L. But the $\dddot{q}/M^2$ term in L is only negligible compared to the $\dot{q}^2$ term when the time derivative is small compared to M but this is not the case for the ω_- solution.

The situation is a bit different if $\eta = -1$ because in this case all of the $\omega_\pm$ can be real provided that $4m^2 \leq M^2$. Once this is violated then the square root becomes imaginary and so $\omega_\pm^2$ becomes complex. Complex $\omega_\pm$ again indicates a solution growing exponentially with time, and so an instability. We see from this example that instability often arises, though the example is simple enough to obscure whether introducing nonlinearities into the equations might change this conclusion. Investigating higher-derivative theories within the Hamiltonian framework provides a more robust diagnostic for potential instabilities.

In order to set up the Hamiltonian formalism for the general Lagrangian (7.1.46) it is useful to reformulate it so that L involves only first time derivatives. This can always be done by adding more variables. Consider therefore the alternative formulation where

$$S[Q_i(t), \lambda(t)] = \int_{t_0}^{t_f} \mathrm{d}\tau\, \widetilde{L} = \int_{t_0}^{t_f} \mathrm{d}\tau \Big[L(Q_1, Q_2, \dot{Q}_2) + \lambda(Q_2 - \dot{Q}_1) \Big] . \tag{7.1.56}$$

In this formulation there are two coordinates, Q_1 and Q_2, but there is also a constraint that forces Q_2 to be the time derivative of Q_1. The variable λ is the Lagrange multiplier whose variation is designed to impose this constraint, along the lines described in §2.5.2 (see also §A.2).

To verify that this reproduces the same equations of motion as does the original formulation using $L(q, \dot{q}, \ddot{q})$ we vary the action (7.1.56), leading to

$$\begin{aligned}\delta S = &\left[\frac{\partial L}{\partial \dot{Q}_2}\,\delta Q_2 - \lambda\,\delta Q_1\right]_{t_0}^{t_f} \\ &+ \int_{t_0}^{t_f} \mathrm{d}\tau \left\{ \left[\frac{\partial L}{\partial Q_1} + \dot{\lambda}\right]\delta Q_1 + \left[\frac{\partial L}{\partial Q_2} + \lambda - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{Q}_2}\right)\right]\delta Q_2 + \left[Q_2 - \dot{Q}_1\right]\delta\lambda \right\}.\end{aligned} \tag{7.1.57}$$

Requiring this to be stationary for arbitrary variations for which δQ_1 and δQ_2 vanish at the endpoints leads to the Euler-Lagrange equations

$$\frac{\partial L}{\partial Q_1} + \dot{\lambda} = 0\,, \quad \lambda = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{Q}_2}\right) - \frac{\partial L}{\partial Q_2} \quad \text{and} \quad Q_2 = \dot{Q}_1\,. \tag{7.1.58}$$

Substituting the last two of these into the first indeed reproduces (7.1.48).

Going to the Hamiltonian formulation is straightforward with these new variables, because for them L depends only on Q_1, Q_2, λ and their first time derivatives. The starting point is to construct the canonical momenta:

$$P_1 := \frac{\partial L}{\partial \dot{Q}_1} = -\lambda\,, \quad P_2 := \frac{\partial L}{\partial \dot{Q}_2} = P_2(Q_1, Q_2, \dot{Q}_2) \quad \text{and} \quad P_\lambda := \frac{\partial L}{\partial \dot{\lambda}} = 0\,, \tag{7.1.59}$$

and we assume that the second of these can be solved to give

$$\dot{Q}_2 = \dot{Q}_2(Q_1, Q_2, P_2)\,. \tag{7.1.60}$$

The equation for P_λ cannot be similarly solved for $\dot{\lambda}$ but this does not really matter because $\dot{\lambda}$ does not appear at all in the action anyway. We cannot solve for $\dot{Q}_1$ just using the definitions of the canonical momenta, but this is also not a problem (as we see below) because $\dot{Q}_1$ drops out once we use the equation for P_1.

The Hamiltonian is then found from the standard expression (7.1.3):

$$\begin{aligned} H &= P_1\dot{Q}_1 + P_2\dot{Q}_2 + P_\lambda\dot{\lambda} - \Big\{L[Q_1, Q_2, \dot{Q}_2(Q_1, Q_2, P_2)] + \lambda(Q_2 - \dot{Q}_1)\Big\} \\ &= (P_1 + \lambda)\dot{Q}_1 + P_2\dot{Q}_2(Q_1, Q_2, P_2) - \lambda Q_2 - L[[Q_1, Q_2, \dot{Q}_2(Q_1, Q_2, P_2)] \\ &= P_2\dot{Q}_2(Q_1, Q_2, P_2) + P_1\,Q_2 - L[Q_1, Q_2, \dot{Q}_2(Q_1, Q_2, P_2)]\,, \end{aligned} \tag{7.1.61}$$

where the second line uses $P_\lambda = 0$ and the final equality eliminates λ using the definition (7.1.59) for P_1.

What is important about this last expression is that it is linear in P_1, regardless of the details of the functional form of L. In the absence of a constraint that restricts the allowed range for P_1 this means that H is not a function that is bounded from below. As mentioned earlier, it is this unboundedness that is ultimately responsible for instability.[23]

[23]This instability is particularly pressing for the quantum theory, where everything that is possible becomes compulsory: quantum transitions can drive the system to lower and lower energies even if the initial conditions did not do so.

7.2 Phase Space and Poisson Brackets

Returning to the main line of development, we wish to further explore how time evolution works within the Hamiltonian framework. The main change from the Lagrangian picture is that for N particles our space of paths is now categorized by $6N$ variables: 3 position and 3 momentum coordinates for each particle. This $6N$-dimensional space with coordinates $\{q^A, p_B\}$ is called the system's *phase space* (as opposed to the *configuration space* spanned by the coordinates $\{q^A\}$ alone), and it provides the stage on which the drama of Hamiltonian mechanics unfolds.

The other main change is that the equations of motion – eqs. (7.1.7), repeated here

$$\dot{q}^A = \frac{\partial H}{\partial p_A} \quad \text{and} \quad \dot{p}_B = \frac{\partial H}{\partial q^B} \tag{7.2.1}$$

are a set of coupled first-order differential equations and so each should be expected to generate one integration constant when solved. These $6N$ integration constants correspond to the two integration constants obtained for each of the $3N$ variables q^A when their second-order differential equations are integrated in the Lagrangian approach. In both formulations $6N$ constants provide enough constants to parameterize the initial conditions $\{q^A(t_0), p_B(t_0)\}$ (or $\{q^A(t_0), \dot{q}^B(t_0)\}$) required to fully specify the evolution.

In general the rate of change of any quantity $F(q,p,t)$ built from the basic variables q^A and p_B is given by

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \left(\frac{\partial F}{\partial q^A}\right)\dot{q}^A + \left(\frac{\partial F}{\partial p_A}\right)\dot{p}_A + \frac{\partial F}{\partial t} = \left(\frac{\partial F}{\partial q^A}\right)\frac{\partial H}{\partial p_A} - \left(\frac{\partial F}{\partial p_A}\right)\frac{\partial H}{\partial q^A} + \frac{\partial F}{\partial t}\,. \tag{7.2.2}$$

This is particularly simple when specialized to $F = H$ since then most terms cancel leaving

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t}\,. \tag{7.2.3}$$

This is how energy conservation emerges in this forulation: H is conserved provided all of its time-dependence comes through the time-dependence of $q(t)$ and $p(t)$, and it does not itself have its own explicit time dependence.

Whether energy is conserved or not can be rephrased in terms of the original Lagrangian because the definition of H implies

$$\left(\frac{\partial H}{\partial t}\right)_{q,p} = -\left(\frac{\partial L}{\partial t}\right)_{q,\dot{q}}\,, \tag{7.2.4}$$

where the subscripts show what is held fixed when taking the partial derivative with respect to t. We see that $\partial L/\partial t = 0$ is required for energy conservation. This is consistent with how energy conservation emerges from Noether's theorem – *c.f.* section §2.3.3 above – as a consequence of time-translation invariance. For it to do so the change of the Lagrangian under a time translation $\delta q^A = \dot{q}^A$ must be a total time derivative of something, and comparing

$$\delta L = \left(\frac{\partial L}{\partial \dot{q}^A}\right)\ddot{q}^A + \left(\frac{\partial L}{\partial q^A}\right)\dot{q}^A \tag{7.2.5}$$

to $\mathrm{d}L/\mathrm{d}t$ shows that $\delta L = \mathrm{d}L/\mathrm{d}t$, but only if $\partial L/\partial t = 0$.

7.2.1 The Poisson bracket

The combination of derivatives appearing in (7.2.2) comes up frequently enough that it has a name; *Poisson bracket*. That is, for any two functions $f(q,p)$ and $g(q,p)$ defined on phase space the Poisson bracket is defined as

$$\Big\{f\,,g\Big\} := \frac{\partial f}{\partial q^A}\,\frac{\partial g}{\partial p_A} - \frac{\partial f}{\partial p_A}\,\frac{\partial g}{\partial q^A}\,, \tag{7.2.6}$$

(with, as usual, there being an implied sum on A). In terms of this the evolution equation (7.2.2) can be written

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \Big\{F\,,H\Big\} + \frac{\partial F}{\partial t}\,. \tag{7.2.7}$$

The Poisson bracket has a number of properties that follow directly from its definition. For any functions $f(q,p)$, $g(q,p)$ and $h(q,p)$ defined on phase space:

$$\begin{aligned} \text{antisymmetry}: &\quad \Big\{f\,,g\Big\} = -\Big\{g\,,f\Big\} \\ \text{bilinearity}: &\quad \Big\{f+g\,,h\Big\} = \Big\{f\,,h\Big\} + \Big\{g\,,h\Big\} \\ \text{product rule}: &\quad \Big\{fg\,,h\Big\} = f\Big\{g\,,h\Big\} + \Big\{f\,,h\Big\}g\,. \end{aligned} \tag{7.2.8}$$

Also,

$$\Big\{f\,,c\Big\} = 0 \quad \text{if } c \text{ is a constant}\,, \tag{7.2.9}$$

and taking a time derivative implies

$$\partial_t\Big\{f\,,g\Big\} = \Big\{\partial_t f\,,g\Big\} + \Big\{f\,,\partial_t g\Big\}\,. \tag{7.2.10}$$

Finally, the Poisson bracket satisfies the following *Jacobi identity*:

$$\Big\{f\,,\Big\{g\,,h\Big\}\Big\} + \Big\{g\,,\Big\{h\,,f\Big\}\Big\} + \Big\{h\,,\Big\{f\,,g\Big\}\Big\} = 0\,. \tag{7.2.11}$$

The Poisson bracket also simplifies if one or both of its arguments is one of the coordinates on phase space:

$$\Big\{f\,,q^A\Big\} = -\frac{\partial f}{\partial p_A} \quad \text{and} \quad \Big\{f\,,p_A\Big\} = \frac{\partial f}{\partial q^A}\,, \tag{7.2.12}$$

and so in particular

$$\Big\{q^A\,,q^B\Big\} = 0\,, \quad \Big\{p_A\,,p_B\Big\} = 0 \quad \text{and} \quad \Big\{q^B\,,p_A\Big\} = \delta^B_A\,. \tag{7.2.13}$$

7.2.2 Symmetries Revisited

We have seen in eqs. (7.2.3) and (7.2.4) that for systems where the Lagrangian does not depend explicitly on time – *i.e.* when $\partial L/\partial t = 0$ – that several related things happen:

1. First, the system described by L is invariant under time translations,
$$\delta q^A = \delta t\, \dot{q}^A \quad \text{and} \quad \delta \dot{q}^A = \delta t\, \ddot{q}^A\,. \tag{7.2.14}$$
This transformation is a symmetry because under it $\delta L = \delta t\,(\mathrm{d}L/\mathrm{d}t)$, which leaves the action $S = \int \mathrm{d}t\; L$ invariant because the change in L is a total time derivative.

2. Second, time-translation invariance implies Noether's theorem guarantees a conservation law – *c.f.* the discussion in §2.3.3 – which in this case is the energy as defined in (2.3.8).

3. Third, the Hamiltonian as defined by (7.1.3) is equal to the energy as defined by (2.3.8), and its conservation is verified explicitly by its equation of motion (7.2.3) that for time-translation invariant systems states $\mathrm{d}H/\mathrm{d}t = 0$.

4. Fourth, for any other quantity $F(q,p)$ defined on phase space that does not depend explicitly on t the change under a time translation is given by $\delta F = \delta t\,(\mathrm{d}F/\mathrm{d}t)$ which, using (7.2.2), can be written:
$$\delta F = \delta t\, \frac{\mathrm{d}F}{\mathrm{d}t} = \delta t \Big\{ F\,, H \Big\}\,. \tag{7.2.15}$$
In particular, for any such F the change in F is given by the Poisson bracket of H with F (and we say the Hamiltonian *generates* time translations).

In this section we see that a similar set of statements also hold for other continuous symmetries, and this provides a much more intuitive identification of the conserved quantities whose existence is guaranteed by Noether's theorem.

To this end consider constant translations of the generalized coordinates, $\delta q^A = \epsilon^A$, for some collection of infinitesimal constants ϵ^A, without yet asking that these be symmetries. We ask whether there exist quantities, call them Q_A, with the property that they generate constant translations for q^A for any function of phase space. Such a quantity must satisfy

$$\delta q^A = \epsilon^A = \epsilon^B \Big\{ q^A\,, Q_B \Big\} \quad \text{and} \quad \delta p_A = 0 = \epsilon^B \Big\{ p_A\,, Q_B \Big\}\,, \tag{7.2.16}$$

for all q^A and p_B.

But inspection of (7.2.13) shows that such a quantity indeed exists: it is given by the canonical momentum for each q^A: $Q_B = p_B$. Furthermore, with this choice the antisymmetric property of (7.2.8) together with the properties (7.2.9) and (7.2.12) automatically ensure

$$\delta F = \epsilon^B \Big\{ F\,, p_B \Big\} = \epsilon^B\, \frac{\partial F}{\partial q^B} \tag{7.2.17}$$

for any function $F(q,p)$ on phase space. This is indeed how we expect an arbitrary function to change under an infinitesimal translation of q^A, as can be seen by Taylor expanding $F(q+\epsilon,p)$ out to linear order in ϵ.

Now suppose this transformation is a symmetry of the system of interest. For the present purposes we take this to mean that the system's Hamiltonian is invariant under this transformation (for a specific choice for the ϵ^A's). That is, when the transformation is a symmetry we have the additional information that

$$\delta H = \epsilon^B \Big\{ H \,, p_B \Big\} = \epsilon^B \, \frac{\partial H}{\partial q^B} = 0 \,. \tag{7.2.18}$$

Ignorable coordinates are special cases of symmetries of this type for which only a single coordinate shifts (such as spatial translations of a centre-of-mass coordinate for an isolated system or a shift of the angular variable ϕ for the two-body central force problem). The upshot is $\{H\,,Q\} = 0$ for the symmetry generator $Q = \epsilon^B p_B$.

Now comes the main point: because the Poisson bracket is antisymmetric – *c.f.* eq. (7.2.8) – there are two ways to read the relation $\{Q\,,H\} = \{H\,,Q\} = 0$. It can either be read (as above) as saying that the transformation generated by Q leaves H unchanged, or it can be read as saying that the transformation generated by H leaves Q unchanged. But the transformation generated by H is just time translation and so having Q be unchanged by this means

$$\Big\{ Q\,,H \Big\} = \frac{\mathrm{d}Q}{\mathrm{d}t} = 0 \,. \tag{7.2.19}$$

That is to say: Q is conserved. This is a particularly transparent way to see why continuous symmetries are related to conservation laws (that is to say, to understand why Noether's theorem is true). It also provides a new way to construct the conserved charge: just ask what quantity generates the symmetry in question. This is very useful because one is rarely lucky enough to set up a problem with all of the symmetry information provided as ignorable coordinates.

Worked example: Generators of Angular Momentum

As an illustration we use this formulation to construct the generators of rotations acting on the position of a particle written using cartesian coordinates $\{x^i\} = \{x,y,z\}$.

We have seen that the matrices that describe rotations about the x, y and z axes are given explicitly by (4.4.1), reproduced here for ease of reference:[24]

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{pmatrix} \,, \quad R_y = \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \quad \text{or} \quad R_z = \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix} \,, \tag{7.2.20}$$

[24] The signs are chosen here to give an active rotation in a right-hand coordinate system, for which a 90^o rotation about the x axis takes $\mathbf{e}_y$ to $\mathbf{e}_z$, a 90^o rotation about the y axis takes $\mathbf{e}_z$ to $\mathbf{e}_x$ and a 90^o rotation about the z axis takes $\mathbf{e}_x$ to $\mathbf{e}_y$.

For infinitesimal rotation angles these become

$$R_x \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\theta_x \\ 0 & \theta_x & 1 \end{pmatrix} , \quad R_y = \begin{pmatrix} 1 & 0 & \theta_y \\ 0 & 1 & 0 \\ -\theta_y & 0 & 1 \end{pmatrix} \quad \text{or} \quad R_z = \begin{pmatrix} 1 & -\theta_z & 0 \\ \theta_z & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{7.2.21}$$

and so imply the transformation rules respectively become

$$\delta_x x = 0 \,, \quad \delta_x y = -\theta_x z \,, \quad \delta_x z = \theta_x y \tag{7.2.22a}$$

$$\delta_y x = \theta_y z \,, \quad \delta_y y = 0 \,, \quad \delta_y z = -\theta_y x \tag{7.2.22b}$$

$$\delta_z x = -\theta_z y \,, \quad \delta_z y = \theta_z x \,, \quad \delta_z z = 0 \,. \tag{7.2.22c}$$

These can be written in terms of Poisson brackets as

$$\delta_x x_i = \theta_x \Big\{ x_i \,, L_x \Big\} \,, \quad \delta_y x_i = \theta_y \Big\{ x_i \,, L_y \Big\} \quad \text{and} \quad \delta_z x_i = \theta_z \Big\{ x_i \,, L_z \Big\} \,, \tag{7.2.23}$$

where

$$L_x = y p_z - z p_y \,, \quad L_y = z p_x - x p_z \quad \text{and} \quad L_z = x p_y - y p_x \tag{7.2.24}$$

(or, more compactly, $L_i = \epsilon_{ijk} x_j p_k$) are the components of angular momentum. The proof of this just repeatedly uses the properties (7.2.8) through (7.2.13). For instance

$$\Big\{ y \,, L_x \Big\} = \Big\{ y \,, y p_z - z p_y \Big\} = y \Big\{ y \,, p_z \Big\} - \Big\{ y \,, z p_y \Big\} = -z \Big\{ y \,, p_y \Big\} = -z \,, \tag{7.2.25}$$

and similarly for the other L_i and other coordinates x_j.

Notice that these rotation generators also do the correct thing when they act on each other, since the components L_i of angular momentum should transform as a vector. Explicitly, we expect

$$\delta_x L_x = 0 \,, \quad \delta_x L_y = -\theta_x L_z \,, \quad \delta_x L_z = \theta_x L_y \tag{7.2.26a}$$

$$\delta_y L_x = \theta_y L_z \,, \quad \delta_y L_y = 0 \,, \quad \delta_y L_z = -\theta_y L_x \tag{7.2.26b}$$

$$\delta_z L_x = -\theta_z L_y \,, \quad \delta_z L_y = \theta_z L_x \,, \quad \delta_z L_z = 0 \,, \tag{7.2.26c}$$

or, equivalently,

$$\delta_j L_i = \epsilon_{ijk} \theta_j L_k \,. \tag{7.2.27}$$

To verify this is true we explicitly compute $\delta_i L_j = \theta_i \{ L_j \,, L_i \}$. For example:

$$\begin{aligned} \Big\{ L_x \,, L_y \Big\} &= \Big\{ y p_z - z p_y \,, z p_x - x p_z \Big\} = \Big\{ y p_z \,, z p_x \Big\} - \Big\{ y p_z \,, x p_z \Big\} - \Big\{ z p_y \,, z p_x \Big\} + \Big\{ z p_y \,, x p_z \Big\} \\ &= y p_x \Big\{ p_z \,, z \Big\} - 0 - 0 + x p_y \Big\{ z \,, p_z \Big\} = -y p_x + x p_y = L_z \,, \end{aligned} \tag{7.2.28}$$

and similarly for the other two possible pairs: $\{L_y \,, L_z\} = L_x$ and $\{L_z \,, L_x\} = L_y$, all of which can be compactly written $\{L_i \,, L_j\} = \epsilon_{ijk} \, L_k$. Using this in $\delta_i L_j = \theta_i \{L_j \,, L_i\}$ then reproduces (7.2.27).

Since the angular momenta generate rotations, the arguments of this section show (again) why angular momentum is conserved in rotationally invariant theories.

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Finally, notice also that if f and g are generators of a symmetry (so $\{f\,,H\} = \{g\,,H\} = 0$) then their Poisson bracket also generates a symmetry, in the sense that $\{\{f\,,g\}\,,H\} = 0$. To prove this use the Jacobi identity (7.2.11) to write

$$\Big\{\Big\{f\,,g\Big\}\,,H\Big\} = \Big\{f\,,\Big\{g\,,H\Big\}\Big\} + \Big\{g\,,\Big\{H\,,f\Big\}\Big\}\,, \tag{7.2.29}$$

from which the conclusion follows. This just expresses the fact that performing two symmetries one after the other is also a symmetry, a fact that plays a role in concluding that the set of symmetries forms a group (in the mathematical sense of the word 'group') and the generators of continuous symmetries – together with the Poisson brackets – form an algebra (called the *Lie algebra* of the continuous symmetry group).

Worked example: Charged particle in a magnetic field (again)

As another illustration of the Poisson bracket consider the point particle with electric charge q moving in a magnetic field $\mathbf{B}$.

In this case the Lagrangian for the system is given by (2.6.22) restricted to the case of a vanishing electrostatic potential $\Phi = 0$:

$$L = \tfrac{1}{2} m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}}\cdot\mathbf{A}\,, \tag{7.2.30}$$

where (as usual) the vector potential $\mathbf{A}$ is defined by $\mathbf{B} = \nabla\times\mathbf{A}$, and as a result is only defined *up to a gauge transformation* $\mathbf{A}\to\mathbf{A}+\nabla\omega$ for an arbitrary function ω.

The canonical momentum is defined as usual by

$$\mathbf{p} = \frac{\partial L}{\partial\dot{\mathbf{r}}} = m\dot{\mathbf{r}} + q\mathbf{A} \quad \text{which inverts to give} \quad \dot{\mathbf{r}} = \frac{1}{m}\Big(\mathbf{p} - q\mathbf{A}\Big)\,, \tag{7.2.31}$$

and so the Hamiltonian becomes

$$H = \mathbf{p}\cdot\dot{\mathbf{r}} - L = \frac{\mathbf{p}}{m}\cdot(\mathbf{p}-q\mathbf{A}) - \frac{1}{2m}(\mathbf{p}-q\mathbf{A})^2 - q\mathbf{A}\cdot(\mathbf{p}-q\mathbf{A}) = \frac{1}{2m}(\mathbf{p}-q\mathbf{A})^2\,. \tag{7.2.32}$$

Although $\mathbf{p}$ is not gauge invariant the Hamiltonian $H = \frac{1}{2}m\dot{\mathbf{r}}^2$ is.

Writing the components in rectangular coordinates, $\mathbf{r} = x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z$ and $\mathbf{p} = p_x\,\mathbf{e}_x + p_y\,,\mathbf{e}_y + p_z\,\mathbf{e}_z$, the Poisson brackets for the canonical variables are the standard ones:

$$\Big\{x_i\,,x_j\Big\} = \Big\{p_i\,,p_j\Big\} = 0 \quad \text{and} \quad \Big\{x_i\,,p_j\Big\} = \delta_{ij}\,. \tag{7.2.33}$$

The Poisson brackets for the velocities then become

$$\Big\{x_i\,,\dot{x}_j\Big\} = \frac{1}{m}\Big\{x_i\,,p_j - qA_j(x)\Big\} = \frac{1}{m}\delta_{ij} \tag{7.2.34}$$

and

$$\Big\{\dot{x}_i\,,\dot{x}_j\Big\} = \frac{1}{m^2}\Big\{p_i - qA_i(x)\,,p_j - qA_j(x)\Big\} = \frac{q}{m^2}\Big(\partial_i A_j - \partial_j A_i\Big) = \frac{q}{m^2}\epsilon_{ijk}B_k\,. \tag{7.2.35}$$

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7.3 Canonical Transformations

Much of the power of Lagrangian methods comes from the ease with which they allow very general changes of variables within configuration space,

$$q^A \to Q^A = Q^A(q)\,, \tag{7.3.1}$$

what is sometimes called a *point* transformation. In these types of transformations the generalized velocities transform in a way that is dictated by the underlying point transformation, with $\dot{q}^A \to \dot{Q}^A = (\partial Q^A/\partial q^B)\,\dot{q}^B$.

There is a much broader freedom to change variables possible within the Hamiltonian framework because it treats the q's and p's as independent of variables with the p's only becoming related to the generalized velocities once Hamilton's equations (7.1.7) are imposed. It is therefore useful to consider general changes of variables on phase space, of the form

$$q^A \to Q^A = Q^A(q,p) \quad \text{and} \quad p_B \to P_B = P_B(q,p)\,, \tag{7.3.2}$$

with both positions and momenta involved from the get-go.

The main thing we ask of such a change of variables is that it not change the physics being studied, and this means that we must require the variational problems $\delta S = 0$ in the two variables to be equivalent. The new system after the change of variables should still be described by a Hamiltonian, call it[25] $K(Q,P)$, such that varying the action built using K produces the same stationary trajectories within phase space as do the stationary curves found using the original Hamiltonian $H(q,p)$. That is, we require

$$\delta \int_{t_0}^{t_f} \mathrm{d}t\, \Big[p_A \dot{q}^A - H(q,p)\Big] = 0 \tag{7.3.3}$$

which implies

$$\dot{q}^A = \frac{\partial H}{\partial p_A} \quad \text{and} \quad \dot{p}_A = -\frac{\partial H}{\partial q^A} \tag{7.3.4}$$

to describe the same phase-space trajectories as

$$\delta \int_{t_0}^{t_f} \mathrm{d}t\, \Big[P_A \dot{Q}^A - K(Q,P)\Big] = 0 \tag{7.3.5}$$

which implies

$$\dot{Q}^A = \frac{\partial K}{\partial P_A} \quad \text{and} \quad \dot{P}_A = -\frac{\partial K}{\partial Q^A}\,. \tag{7.3.6}$$

As discussed in §2.4, this condition is true if the integrands of the two formulations of the action differ only by the time derivative, $\mathrm{d}F/\mathrm{d}t$ of some function F. Written as differentials this says

$$p_A\,\mathrm{d}q^A - H(q,p)\,\mathrm{d}t = P_A\,\mathrm{d}Q^A - K(Q,P)\,\mathrm{d}t + \mathrm{d}F\,. \tag{7.3.7}$$

[25] The Hamiltonians H and K can also depend explicitly on t but we suppress writing this argument for notational clarity.

Any transformation that satisfies this condition is called a *canonical transformation*. Rewriting (7.3.7) as[26]

$$\mathrm{d}F_1 = p_A\,\mathrm{d}q^A - P_A\,\mathrm{d}Q^A + \Big[K(Q,P) - H(q,p)\Big]\,\mathrm{d}t\,, \tag{7.3.8}$$

shows that it is natural to regard $F = F_1(q,Q,t)$ to be a function of q^A and Q^B, in which case (7.3.8) implies

$$\left(\frac{\partial F_1}{\partial q^A}\right)_{Q,t} = p_A\,, \quad \left(\frac{\partial F_1}{\partial Q^A}\right)_{q,t} = -P_A \tag{7.3.9}$$

and

$$\left(\frac{\partial F_1}{\partial t}\right)_{q,Q} = K(Q,P) - H(q,p)\,. \tag{7.3.10}$$

The function $F_1(q,Q,t)$ is called the *generating function* of the canonical transformation since the transformation is completely specified once $F_1(q,Q,t)$ is given. That is, given $H(q,p,t)$ and $F_1(q,Q,t)$ the new canonical positions $Q^A(q,p)$ are defined by solving the equations $p_A(q,Q) = \partial F_1/\partial q^A$ for the Q^A's. The new momenta are defined by substituting the result for $Q^A(q,p)$ into $P^A(q,p) = -\partial F_1/\partial Q^A$. The Hamiltonian in the new variables is then found using $K = H + \partial_t F_1$ where $p_A(Q,P)$ and $q^B(Q,P)$ are found by inverting the expressions $Q^A(q,p)$ and $P_B(q,p)$. A useful consequence of the existence of a generating function – and in particular the relation (7.3.9) – is the further identity

$$\frac{\partial P_A}{\partial q^B} = -\frac{\partial^2 F_1}{\partial q^B \partial Q^A} = -\frac{\partial^2 F_1}{\partial Q^A \partial q^B} = -\frac{\partial p_B}{\partial Q^A}\,. \tag{7.3.11}$$

It is also possible to define a generating functional $F_2(q,P)$ that has q^A and P_B as its natural variables. This is found by defining $F_2(q,P,t) = F_1 + Q^A P_A$ since then $\mathrm{d}F_2 = \mathrm{d}F_1 + Q^A \mathrm{d}P_A + P_A \mathrm{d}Q^A$, which together with (7.3.8) implies

$$\mathrm{d}F_2 = p_A\,\mathrm{d}q^A + Q^A\,\mathrm{d}P_A + \Big[K(Q,P) - H(q,p)\Big]\,\mathrm{d}t\,, \tag{7.3.12}$$

and so

$$\left(\frac{\partial F_2}{\partial q^A}\right)_{P,t} = p_A\,, \quad \left(\frac{\partial F_2}{\partial P_A}\right)_{q,t} = Q^A \tag{7.3.13}$$

and

$$\left(\frac{\partial F_2}{\partial t}\right)_{q,P} = K(Q,P) - H(q,p)\,. \tag{7.3.14}$$

Once $F_2(q,P,t)$ is specified $P_A(q,p)$ is found by solving $\partial F_2/\partial q^A = p_A$ for P_A. Using this result in $Q^A = \partial F_2/\partial P_A$ then gives $Q^A(q,p)$, and the new Hamiltonian is obtained from $K = H + \partial_t F_2$. A similar procedure defines a generating function $F_3(p,Q,t) = F_1 - p_A q^A$ and $F_4(p,P,t) = F_2 - p_A q^A$, should these be desired. These other forms for the generating function allow (7.3.11) to be generalized to

$$\frac{\partial Q^A}{\partial q^B} = \frac{\partial^2 F_2}{\partial q^B \partial P_A} = \frac{\partial^2 F_2}{\partial P_A \partial q^B} = \frac{\partial p_B}{\partial P_A}\,, \tag{7.3.15}$$

[26] The subscript '1' is added to F to distinguish it from the other generating functions defined below.

$$\frac{\partial P_A}{\partial p_B} = -\frac{\partial^2 F_3}{\partial p_B \partial Q^A} = -\frac{\partial^2 F_3}{\partial Q^A \partial p_B} = \frac{\partial q^B}{\partial Q^A}\,, \tag{7.3.16}$$

and

$$\frac{\partial Q^A}{\partial p_B} = \frac{\partial^2 F_4}{\partial p_B \partial P_A} = \frac{\partial^2 F_4}{\partial P_A \partial p_B} = -\frac{\partial q^B}{\partial P_A}\,. \tag{7.3.17}$$

7.3.1 Preservation of Poisson brackets

Because a canonical transformation preserves the Hamiltonian structure of the equations of motion it also follows that it preserves the Poisson brackets, in the sense described in this section.

Consider two functions, u and w, on phase space and a canonical transformation $Q = Q(q,p)$ and $P = P(q,p)$. We have two independent ways to write u and w: either $u = U(Q,P)$ and $w = W(Q,P)$ or $u = u(q,p)$ and $w = w(q,p)$ where

$$u(q,p) = U[Q(q,p), P(q,p)] \quad \text{and} \quad w(q,p) = W[Q(q,p), P(q,p)]\,. \tag{7.3.18}$$

This means there are two natural ways to define the Poisson bracket between u and w, depending on whether the derivatives are taken with respect to (q,p) or (Q,P):

$$\Big\{u\,,w\Big\}_{qp} := \frac{\partial u}{\partial q^A}\frac{\partial w}{\partial p_A} - \frac{\partial u}{\partial p_A}\frac{\partial w}{\partial q^A} \quad \text{or} \quad \Big\{u\,,w\Big\}_{QP} = \frac{\partial U}{\partial Q^A}\frac{\partial W}{\partial P_A} - \frac{\partial U}{\partial P_A}\frac{\partial W}{\partial Q^A}\,. \tag{7.3.19}$$

The rest of this section shows that if $(q,p) \to (Q,P)$ is a canonical transformation then it follows that these two notions of Poisson bracket are the same

$$\Big\{u\,,w\Big\}_{qp} = \Big\{u\,,w\Big\}_{QP}\,, \tag{7.3.20}$$

so there is no need for the subscripts qp or QP.

The proof of this result just diligently uses the definitions. Starting with (7.3.18) shows

$$\frac{\partial u}{\partial q^A} = \frac{\partial U}{\partial Q^B}\frac{\partial Q^B}{\partial q^A} + \frac{\partial U}{\partial P_B}\frac{\partial P_B}{\partial q^A} \quad \text{and} \quad \frac{\partial u}{\partial p_A} = \frac{\partial U}{\partial Q^B}\frac{\partial Q^B}{\partial p_A} + \frac{\partial U}{\partial P_B}\frac{\partial P_B}{\partial p_A}\,, \tag{7.3.21}$$

and similarly for the derivatives of w. Combining these then shows

$$\begin{aligned}\Big\{u\,,w\Big\}_{qp} &= \frac{\partial u}{\partial p_A}\frac{\partial w}{\partial q^A} - \frac{\partial u}{\partial q^A}\frac{\partial w}{\partial p_A} \\ &= \left[\frac{\partial U}{\partial Q^B}\frac{\partial Q^B}{\partial p_A} + \frac{\partial U}{\partial P_B}\frac{\partial P_B}{\partial p_A}\right]\left[\frac{\partial W}{\partial Q^C}\frac{\partial Q^C}{\partial q^A} + \frac{\partial W}{\partial P_C}\frac{\partial P_C}{\partial q^A}\right] \\ &\qquad - \left[\frac{\partial U}{\partial Q^B}\frac{\partial Q^B}{\partial q^A} + \frac{\partial U}{\partial P_B}\frac{\partial P_B}{\partial q^A}\right]\left[\frac{\partial W}{\partial Q^C}\frac{\partial Q^C}{\partial p_A} + \frac{\partial W}{\partial P_C}\frac{\partial P_C}{\partial p_A}\right] \\ &= \left[\frac{\partial U}{\partial P_B}\frac{\partial W}{\partial Q^C} - \frac{\partial U}{\partial Q^C}\frac{\partial W}{\partial P_B}\right]\left[\frac{\partial P_B}{\partial p_A}\frac{\partial Q^C}{\partial q^A} - \frac{\partial P_B}{\partial q^A}\frac{\partial Q^C}{\partial p_A}\right] \\ &\quad + \frac{\partial U}{\partial P_B}\frac{\partial W}{\partial P_C}\left[\frac{\partial P_B}{\partial p_A}\frac{\partial P_C}{\partial q^A} - \frac{\partial P_B}{\partial q^A}\frac{\partial P_C}{\partial p_A}\right] + \frac{\partial U}{\partial Q^B}\frac{\partial W}{\partial Q^C}\left[\frac{\partial Q^B}{\partial p_A}\frac{\partial Q^C}{\partial q^A} - \frac{\partial Q^B}{\partial q^A}\frac{\partial Q^C}{\partial p_A}\right]\,.\end{aligned} \tag{7.3.22}$$

So far we have not used that the relation $(q,p) \to (Q,P)$ is a canonical transformation. In particular, being canonical means eq. (7.3.11) and eqs. (7.3.15) through (7.3.17) all hold:

$$\frac{\partial P_A}{\partial q^B} = -\frac{\partial p_B}{\partial Q^A}\,, \quad \frac{\partial Q^A}{\partial q^B} = \frac{\partial p_B}{\partial P_A}\,, \quad \frac{\partial P_A}{\partial p_B} = \frac{\partial q^B}{\partial Q^A} \quad \text{and} \quad \frac{\partial Q^A}{\partial p_B} = -\frac{\partial q^B}{\partial P_A}\,. \tag{7.3.23}$$

These in turn imply

$$\frac{\partial P_B}{\partial p_A}\frac{\partial P_C}{\partial q^A} - \frac{\partial P_B}{\partial q^A}\frac{\partial P_C}{\partial p_A} = \frac{\partial q^A}{\partial Q^B}\frac{\partial P_C}{\partial q^A} + \frac{\partial p_A}{\partial Q^B}\frac{\partial P_C}{\partial p_A} = \frac{\partial P_C}{\partial Q^B} = 0\,, \tag{7.3.24}$$

where the second-last equality uses the chain rule for differentiation. Similarly

$$\frac{\partial Q^B}{\partial p_A}\frac{\partial Q^C}{\partial q^A} - \frac{\partial Q^B}{\partial q^A}\frac{\partial Q^C}{\partial p_A} = \frac{\partial Q^B}{\partial p_A}\frac{\partial p_A}{\partial P_C} + \frac{\partial Q^B}{\partial q^A}\frac{\partial q^A}{\partial P_C} = \frac{\partial Q^B}{\partial P_C} = 0\,, \tag{7.3.25}$$

and

$$\frac{\partial P_B}{\partial p_A}\frac{\partial Q^C}{\partial q^A} - \frac{\partial P_B}{\partial q^A}\frac{\partial Q^C}{\partial p_A} = \frac{\partial P_B}{\partial p_A}\frac{\partial p_A}{\partial P_C} + \frac{\partial P_B}{\partial q^A}\frac{\partial q^A}{\partial P_C} = \frac{\partial P_B}{\partial P_C} = \delta^C_B\,. \tag{7.3.26}$$

Using these in (7.3.22) finally gives

$$\Big\{u\,, w\Big\}_{qp} = \frac{\partial U}{\partial P_B}\frac{\partial W}{\partial Q^B} - \frac{\partial U}{\partial Q^B}\frac{\partial W}{\partial P_B} = \Big\{u\,, w\Big\}_{QP}\,, \tag{7.3.27}$$

as claimed.

7.3.2 Examples of Canonical Tranformations

This section tries to make the above discussion more concrete by exploring simple examples of canonical transformations.

Point transformations

Consider first the case where the generating function $F_2(q,P,t)$ is linear in the new momenta:

$$F_2(q,P,t) = f^A(q,t)\,P_A\,, \tag{7.3.28}$$

where $f^A(q,t)$ are a collection of arbitrary functions of the generalized positions q^A. In this case eqs. (7.3.13) become

$$p_A = \frac{\partial F_2}{\partial q^A} = P_B\frac{\partial f^B}{\partial q^A} \quad \text{and} \quad Q^A = \frac{\partial F_2}{\partial P_A} = f^A(q,t)\,. \tag{7.3.29}$$

The second of these equations shows the generalized coordinates are shuffled in an arbitrary way and so this transformation is the point transformation $q^A \to Q^A = f^A(q,t)$ considered in earlier sections when discussing Lagrangian methods. The first of eqs. (7.3.29) can be solved for P_A provided the Jacobian matrix $\partial f^B/\partial q^A$ is invertible (as is required if the transformation from the q's to the Q's is to be invertible), and it simply ensures that the momenta transform as would follow from the definitions $p_A = \partial L/\partial \dot{q}^A$ and $P_A = \partial L/\partial \dot{Q}^A$.

This shows that arbitrary invertible point transformations, $q^A \to Q^A(q,t)$, are also canonical transformations. If f^A is time independent then (7.3.14) shows that the old and new Hamiltonians H and K are related by simply performing the change of variables:

$$K(Q,P) = H[q(Q,P), p(Q,P)]\,. \tag{7.3.30}$$

The trivial (or 'identity') transformation is a special case of this transformation for which $f^A(q) = q^A$ and so $F_2(q,P) = q^A P_A$. In this case eqs. (7.3.29) become $p_A = P_A$ and $Q^A = q^A$, as expected for the identity transformation.

The harmonic oscillator

A more complicated example applies to the 1D simple harmonic oscillator, whose initial coordinate is $q(t)$ and whose Hamiltonian is as given in (7.1.10), which we write as

$$H(q,p) = \frac{p^2}{2m} + \frac{m\omega^2}{2}\, q^2\,, \tag{7.3.31}$$

where $\omega^2 = k/m$ is the oscillator frequency. Hamilton's equations in these variables are the usual oscillator equations

$$\dot q = \frac{p}{m} \quad \text{and} \quad \dot p = m\ddot q = -m\omega^2 q\,. \tag{7.3.32}$$

Consider in this case a canonical transformation generated by the generating function

$$F_1(q,Q) = \tfrac12 m\omega q^2 \cot Q\,. \tag{7.3.33}$$

With this choice eqs. (7.3.9) become

$$p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q \quad \text{and} \quad P = -\frac{\partial F_1}{\partial Q} = \tfrac12 m\omega q^2 \csc^2 Q\,. \tag{7.3.34}$$

Solving the second of these for q gives the explicit form for the resulting canonical transformation:

$$q = \sqrt{\frac{2P}{m\omega}}\, \sin Q \quad \text{and} \quad p = \sqrt{2m\omega P}\, \cos Q\,, \tag{7.3.35}$$

which clearly mixes up positions and momenta in a nontrivial way.

This particular canonical transformation is noteworthy because the Hamiltonian is particularly simple once written in the new variables. Because $\partial_t F_1 = 0$ the new Hamiltonian is

$$K(Q,P) = H[q(Q,P), p(Q,P)] = \omega P \cos^2 Q + \omega P \sin^2 Q = \omega P\,. \tag{7.3.36}$$

Hamilton's equations (7.3.6) for the new variables are particularly simple:

$$\dot Q = \frac{\partial K}{\partial P} = \omega \quad \text{and} \quad \dot P = -\frac{\partial K}{\partial Q} = 0\,. \tag{7.3.37}$$

The transformation has had the effect of changing q into an ignorable coordinate, and the equations of motion are solved by

$$Q(t) = Q_0 + \omega\, t \quad \text{and} \quad P(t) = P_0\,, \tag{7.3.38}$$

where Q_0 and P_0 are integration constants. Using these in eqs. (7.3.35) translates this solution to the original variables

$$q = \sqrt{\frac{2P_0}{m\omega}}\; \sin(Q_0 + \omega\, t) \quad \text{and} \quad p = \sqrt{2m\omega P_0}\; \cos(Q_0 + \omega\, t)\,, \tag{7.3.39}$$

which is indeed the usual solution to (7.3.32).

This shows the power of being able to change variables in a way that mixes up positions and momenta: the right choice for a canonical transformation can be general enough to allow a complete solution to the evolution of a particular problem. This leads to some obvious questions, like when is such a transformation possible? And how does one find the magic variable change when it is?

7.3.3 Symplectic manifolds

In many ways the invariance of the Hamiltonian formulation of Newton's laws under arbitrary canonical transformations taken together with the central role played by the Poisson brackets defines the essence of classical mechanics. Many modern treatments describe classical mechanics in those terms: as the study of the geometry of *symplectic manifolds*, allowing the use of powerful tools originally designed to study problems in differential geometry.

Differential geometry is a natural language to use when formulating laws that remain invariant under changes of coordinates. When studying gravitational physics spacetime is regarded as a differential manifold and the focus is on understanding the properties of the metric tensor on that manifold, where the metric is a symmetric rank-two tensor, $g_{\mu\nu}(x) = g_{\nu\mu}(x)$, that defines inner products and distances on the manifold through formulae like

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu\,. \tag{7.3.40}$$

Here $\mathrm{d}x^\mu = \{\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z\}$ denotes a small coordinate displacement and $\mathrm{d}s$ represents the physical invariant distance between the points separated by this coordinate displacement. One studies how $g_{\mu\nu}$ changes under changes of coordinates and identifies quantities like curvature that characterize the geometry of the manifold in a coordinate-invariant way.

A similar approach can be taken to classical mechanics where the starting point is some even-dimensional manifold with coordinates y^μ so that it is phase space that is the differential manifold. The role of the metric is played by an antisymmetric tensor, $\omega_{\mu\nu}(y) = -\omega_{\nu\mu}(y)$, defined on this manifold. This form defines the Poisson bracket structure on the manifold, with the Poisson bracket of two functions $f(y)$ and $g(y)$ defined by

$$\Big\{ f\,, g \Big\} = \omega^{\mu\nu} \frac{\partial f}{\partial y^\mu}\, \frac{\partial g}{\partial y^\nu}\,. \tag{7.3.41}$$

$\omega^{\mu\nu}$ is called a *symplectic form*.

Performing a coordinate transformation $y \to \tilde{y}(y)$ and using the chain rule implies

$$\Big\{ f \,, g \Big\} = \tilde{\omega}^{\mu\nu} \frac{\partial f}{\partial \tilde{y}^\mu} \, \frac{\partial g}{\partial \tilde{y}^\nu} \,. \tag{7.3.42}$$

where

$$\tilde{\omega}^{\alpha\beta} = \omega^{\mu\nu} \, \frac{\partial \tilde{y}^\alpha}{\partial y^\mu} \, \frac{\partial \tilde{y}^\beta}{\partial y^\nu} \,. \tag{7.3.43}$$

This is just the standard transformation rule for any rank-two contravariant tensor $\omega^{\mu\nu}$.

If we define a canonical transformation as one that does not change the Poisson brackets we see its Jacobian $J^\alpha_\mu = \partial \tilde{y}^\alpha / \partial y^\mu$ must satisfy

$$\tilde{\omega}^{\alpha\beta} = \omega^{\mu\nu} \, \frac{\partial \tilde{y}^\alpha}{\partial y^\mu} \, \frac{\partial \tilde{y}^\beta}{\partial y^\nu} = \omega^{\alpha\beta} \,. \tag{7.3.44}$$

That is, $\omega^{\alpha\beta}$ is an *invariant* tensor (as opposed to just transforming *covariantly*).

For any such a form at a given point one can use the freedom to change coordinates to define a special set of coordinates $\{y^\mu\} = \{q^A \,, p_B\}$ for which ω can be written as the specific matrix

$$\omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \,, \tag{7.3.45}$$

where the 0's here represent $n \times n$ zero matrices and I represents the $n \times n$ unit matrix. This is how the labels q^A and p_B arise within this general manifold formulation of phase space.

To make contact between (7.3.44) and earlier sections suppose we write out the Jacobian for a change of variables $\{y^\mu\} = \{q^A \,, p_B\}$ to $\{\tilde{y}^\mu\} = \{Q^A \,, P_B\}$ explicitly:

$$\frac{\partial \tilde{y}^\mu}{\partial y^\nu} = \begin{pmatrix} \partial Q^A / \partial q^B & \partial Q^A / \partial p_B \\ \partial P_A / \partial q^B & \partial P_A / \partial p_B \end{pmatrix} \,, \tag{7.3.46}$$

Using this to write out the right-hand side of (7.3.43) explicitly, with $\omega^{\mu\nu}$ given by (7.3.45) then gives

$$\tilde{\omega}^{\alpha\beta} = \omega^{\mu\nu} \, \frac{\partial \tilde{y}^\alpha}{\partial y^\mu} \, \frac{\partial \tilde{y}^\beta}{\partial y^\nu} = \begin{pmatrix} \{Q^A \,, Q^B\} & \{Q^A \,, P_B\} \\ \{P_A \,, Q^B\} & \{P_A \,, P_B\} \end{pmatrix} \,, \tag{7.3.47}$$

from which we can see that the transformation is canonical – *i.e.* the left-hand side is again given by (7.3.45) – if and only if the new variables satisfy $\{Q^A \,, Q^B\} = \{P_A \,, P_B\} = 0$ and $\{Q^A \,, P_B\} = \delta^A_B$, as found earlier.

Consider next a one-parameter canonical transformation, $Q^A = Q^A(q, p, u)$ and $P_A = P_A(q, p, u)$, for some parameter u, with $u = 0$ corresponding to the identity transformation. Then if we choose $u = \epsilon$ to be infinitesimal we can write

$$Q^A = q^A + \epsilon \, F^A(q, p) \quad \text{and} \quad P_A = p_A + \epsilon \, E_A(q, p) \tag{7.3.48}$$

for some functions F^A and E_A. Putting this into the Jacobian (7.3.46) then gives

$$\partial\tilde{y}^\mu/\partial y^\nu = \begin{pmatrix} \delta^A_B + \epsilon\,\partial F^A/\partial q^B & \epsilon\,\partial F^A/\partial p_B \\ \epsilon\,\partial E_A/\partial q^B & \delta^B_A + \epsilon\,\partial E_A/\partial p_B \end{pmatrix}. \tag{7.3.49}$$

Using this to evaluate (7.3.47) then shows that the condition for (7.3.48) to be canonical at linear order in ϵ is

$$\frac{\partial F^A}{\partial q^B} = -\frac{\partial E_B}{\partial p_A}. \tag{7.3.50}$$

This implies there is locally a function $G(q,p)$ such that

$$F^A = \frac{\partial G}{\partial p_A} \quad \text{and} \quad E_A = -\frac{\partial G}{\partial q^A}, \tag{7.3.51}$$

in which case (7.3.50) is a consequence of the symmetry of double derivatives: $\partial^2 G/\partial q^A \partial p_B = \partial^2 G/\partial p_B \partial q^A$. We say G *generates* the one-parameter family of canonical transformations.

Since the above argument could have been made starting at any point, another way to think of the one parameter family of transformations is as an 'active' transformation rather than a 'passive' one: *i.e.* as the motion of points in phase space rather than a change of description of fixed points. In this alternate description the expression $Q^A = Q^A(q,p,u)$ and $P_A = P_A(q,p,u)$, describe a family of curves in phase space along which any particular point changes. The tangent to these curves

$$\frac{\mathrm{d}Q^A}{\mathrm{d}u} = \frac{\partial G}{\partial p_A} \quad \text{and} \quad \frac{\mathrm{d}P_A}{\mathrm{d}u} = -\frac{\partial G}{\partial q^A}, \tag{7.3.52}$$

are determined by the function $G(q,p)$. In a general set of coordinates this would be written

$$\frac{\mathrm{d}y^\mu}{\mathrm{d}u} = \omega^{\mu\nu}\,\frac{\partial G}{\partial y^\nu}. \tag{7.3.53}$$

This is all very reminiscent of Hamilton's equations, since the classical equations of motion (7.1.7) have precisely the same form as (7.3.52) where the Hamiltonian $H(q,p)$ plays the role of the generator $G(q,p)$ (underlining the role of the Hamiltonian as the generator of time translations). On one hand *any* one-parameter canonical transformation can be regarded as a *Hamiltonian flow* for some choice of function $G(q,p)$. On the other hand this also shows that actual time evolution can be regarded as being a particular case of a canonical transformation.

In either case eqs. (7.3.53) are very geometrical: canonical transformations (and time evolution) are integral curves on the phase-space manifold whose tangents are determined in terms of the gradients of a particular scalar function, with the special structure of classical mechanics arising due to the properties of $\omega_{\mu\nu}(y)$. A full discussion of this goes well beyond the scope of these notes, but the interested reader can consult texts such as those written by Arnold or by Abraham and Marsden.

Worked example: Spin (phase-space as a 2-sphere)

As a nontrivial example of a phase space manifold that is not simply a plane consider the phase space for the precession of a classical spin. This is described by the spherical angles (θ, ϕ) that label the point on a 2-sphere towards which the spin points at any given instant.

We take the action for this spin to be

$$S = \int_{t_0}^{t_f} \mathrm{d}t \; a\,\dot{\phi}\,\cos\theta\,, \tag{7.3.54}$$

where a is a constant. This is unusual in that it is linear in time derivatives, but this is to be expected for a slowly moving spin system for the following reasons. For slowly moving systems the terms with the fewest derivatives dominate and for systems that are invariant under time reversal, $t \to -t$, the leading term involving time derivatives must involve at least two of them and so is quadratic in the velocities. But since spins are not time-reversal invariant their Lagrangian can involve single powers of velocity and so their slow motion is described by an action linear in velocities, like (7.3.54).

The canonical momentum for ϕ in this case is

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = a\,\cos\theta\,, \tag{7.3.55}$$

which suggests we should think of θ as the canonical momentum for ϕ. This is reinforced by the fact that no derivatives of θ appear in S, since this implies that if we had regarded θ as being an independent coordinate then its canonical momentum $p_\theta = \partial L/\partial\dot{\theta}$ would be zero. Put differently: if we regard both θ and ϕ as independent coordinates then we find the system is constrained. The constraint both states that p_θ vanishes and that θ is related to p_ϕ by (7.3.55).

Taking ϕ and $p_\phi = a\cos\theta$ to be the canonical variables, the Hamiltonian for this system is

$$H(\theta,\phi) = p_\phi\dot{\phi} - L = 0\,. \tag{7.3.56}$$

This simply expresses that there is no energy cost for the direction of the spin to change. Nonetheless there is nontrivial dynamics because the Poisson brackets are nontrivial.

Writing $\mathrm{d}p_\phi = -a\,\sin\theta\,\mathrm{d}\theta$ we see that the Poisson brackets in this system for phase-space variables $f(\theta,\phi)$ and $g(\theta,\phi)$ are given by

$$\Big\{ f\,, g \Big\} = \frac{\partial f}{\partial\phi}\,\frac{\partial g}{\partial p_\phi} - \frac{\partial f}{\partial p_\phi}\,\frac{\partial g}{\partial \phi} = -\frac{1}{a\sin\theta}\left(\frac{\partial f}{\partial\phi}\,\frac{\partial g}{\partial\theta} - \frac{\partial f}{\partial\theta}\,\frac{\partial g}{\partial\phi}\right)\,, \tag{7.3.57}$$

and so in particular $\{\phi\,, p_\phi\} = \{\phi\,, a\,\cos\theta\} = 1$ (as it should). Although there is no energy cost for the spin changing there is also no force urging such a change. Because $H = 0$ it is trivial to conclude the equations of motion are

$$\dot{\theta} = \Big\{\theta\,, H\Big\} = 0 \quad \text{and} \quad \dot{\phi} = \Big\{\phi\,, H\Big\} = 0\,. \tag{7.3.58}$$

The rotation invariance of the problem can be shown by considering the following three generators

$$J_1 = a\,\sin\theta\cos\phi\,, \quad J_2 = a\,\sin\theta\sin\phi \quad \text{and} \quad J_3 = a\,\cos\theta\,. \tag{7.3.59}$$

These are just the cartesian components of a vector $\mathbf{J}$ that points in the direction (θ,ϕ) with length a. As is easily checked, the Poisson brackets of these components with one another are given by

$$\begin{aligned}\Big\{J_1\,, J_2\Big\} &= a^2\Big\{\sin\theta\,\cos\phi\,, \sin\theta\,\sin\phi\Big\} \\ &= -\frac{a^2}{a\sin\theta}\Big[(-\sin\theta\,\sin\phi)(\cos\theta\,\sin\phi) - (\cos\theta\,\cos\phi)(\sin\theta\,\cos\phi)\Big] \\ &= a\cos\theta = J_3\,,\end{aligned} \tag{7.3.60}$$

and similarly $\{J_2\,, J_3\} = J_1$ and $\{J_3\,, J_1\} = J_2$. More compactly:

$$\Big\{J_i\,, J_j\Big\} = \epsilon_{ijk}\, J_k\,, \tag{7.3.61}$$

in agreement with the Poisson brackets for the angular momentum generators as given *e.g.* in and just below eq. (7.2.28).

We see that the J_i indeed generates rotations on vectors and so can play the role of angular momentum. Since it is not associated with $\mathbf{r} \times \mathbf{p}$ for a moving degree of freedom it can be regarded as an object's intrinsic spin. With this interpretation the magnitude of total angular momentum (or spin) J is $J^2 = \mathbf{J} \cdot \mathbf{J} = a^2$. Eqs. (7.3.59) then show that the dynamics of θ and ϕ describes how the direction in which the angular momentum points changes with time.

This dynamics can be made more interesting if the spin couples to something, such as to a magnetic field through a magnetic moment coupling of the form $V = -\mu\,\mathbf{B} \cdot \mathbf{J}$, where μ is a constant representing the system's magnetic moment. The sign of V ensures the energy is lowered when the magnetic field and the spin are aligned.

In this case the action replaces (7.3.54) with

$$S = \int_{t_0}^{t_f} \mathrm{d}t \Big[a\,\dot{\phi}\,\cos\theta + \mu a(B_1 \sin\theta\,\cos\phi + B_2 \sin\theta\,\sin\phi + B_3\cos\theta)\Big]\,, \tag{7.3.62}$$

and so the expression for p_ϕ remains unchanged but now the Hamiltonian becomes

$$H = p_\phi\dot{\phi} - L = -\mu a(B_1 \sin\theta\,\cos\phi + B_2\sin\theta\,\sin\phi + B_3\cos\theta) = -\mu\,\mathbf{B}\cdot\mathbf{J}\,. \tag{7.3.63}$$

It is convenient to choose the z-axis parallel to $\mathbf{B}$, in which case $H = -\mu a\, B\cos\theta$. With this Hamiltonian the equations of motion are

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \quad \text{and} \quad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = -\frac{1}{a\sin\theta}\,\frac{\partial H}{\partial \theta} = \mu B\,. \tag{7.3.64}$$

The equations of motion can also be recast directly in terms of $\mathbf{J}$, in terms of which they are

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = \Big\{J_i\,, H\Big\} = -\mu\,\Big\{J_i\,, J_k\Big\}\,B_k = -\mu\,\epsilon_{ikl}B_kJ_l \tag{7.3.65}$$

and so

$$\dot{\mathbf{J}} = -\mu\,\mathbf{B}\times\mathbf{J}\,. \tag{7.3.66}$$

In particular $\mathrm{d}(J^2)/\mathrm{d}t = 2\mathbf{J}\cdot\dot{\mathbf{J}} = 0$ so only the direction of the angular momentum changes. $\dot{\mathbf{J}}$ is also perpendicular to $\mathbf{B}$ and so the instantaneous change to $\mathbf{J}$ is perpendicular to the plane spanned by $\mathbf{J}$ and $\mathbf{B}$: the angular momentum *precesses* around the magnetic field direction.

The precession rate is found by choosing the z-axis parallel to $\mathbf{B}$ so $B_1 = B_2 = 0$ and $B_3 = B = |\mathbf{B}|$. The equations of motion (7.3.65) then imply $\dot{J}_3 = 0$ and so $J_3 = a\cos\theta$ is time-independent (as must also be θ). The evolution of J_1 and J_2 then is

$$\dot{J}_1 = \mu B\, J_2 \quad \text{and} \quad \dot{J}_2 = -\mu B\, J_1\,, \tag{7.3.67}$$

which integrates to give $J_1 = a\sin\theta\cos(\omega\, t+\phi_0)$ and $J_2 = a\sin\theta\sin(\omega\, t+\phi_0)$ and so $\phi = \omega\, t+\phi_0$ where the angular speed of precession is $\omega = \mu B$. Here both ϕ_0 and θ are integration constants determined by the initial conditions.

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7.4 Hamilton-Jacobi methods*

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7.5 Connection to Quantum Mechanics

For many physicists Hamiltonians are first encountered in quantum mechanics rather than classical mechanics and indeed the advent of quantum mechanics is one of the reasons why Hamiltonian methods are so widely used today. This section touches just briefly on how Hamiltonian methods facilitate the transition from classical to quantum mechanics and on how quantum physics provides insights into the foundations of classical mechanics (including explaining why variational principles play so large a role).

7.5.1 Commutation relations

The most obvious connection between classical and quantum methods is the correspondence between the classical Poisson bracket, $\{u\,,w\}$ and the quantum commutator: $[\hat{u}\,,\hat{w}] = \hat{u}\,\hat{w} - \hat{w}\,\hat{u}$ that can be nonzero for the operators that represent physical observables in quantum mechanics.

A fundamental role is played in the quantum version of a theory by the commutation relations that state how the operators representing generalized positions and momenta fail to commute:

$$\Big[\hat{q}^A\,,\hat{q}^B\Big] = \Big[\hat{p}_A\,,\hat{p}_B\Big] = 0 \quad \text{and} \quad \Big[q^A,\,p_B\Big] = i\hbar\,\delta^A{}_B\,. \tag{7.5.1}$$

It is hard to miss the similarity between these expressions and the Poisson brackets given in (7.2.13) that hold between the q's and p's. This underlies the well-known 'quantization' rule in which the operators describing various physical objects are promoted from classical variables to noncommuting quantum ones, subject to the rule that the Poisson bracket in the classical theory goes over to the commutator in the quantum theory:

$$\Big\{A\,,B\Big\} \to \frac{1}{i\hbar}\Big[\hat{A}\,,\hat{B}\Big]\,. \tag{7.5.2}$$

This type of rule is very powerful because it can be applied to *any* system for which a Lagrangian is known. Historically this turned out to be very useful when exploring quantum field theories like electromagnetism and general relativity, for which both classical Lagrangian and Hamiltonian descriptions are well developed (see §10).

7.5.2 Path integrals

There is another, deeper, connection between classical and quantum mechanics. It is deeper in the sense that it provides answers to the question of *why* variational methods play such a central role in classical mechanics and why it is the action that is the thing that should be varied. This other connection comes from what is called the *path integral* formulation of quantum mechanics.

Although it goes beyond the scope of these notes, this section provides a cartoon of how this happens. This section assumes a working knowledge of quantum mechanics, at the level presented in undergraduate classes. We start by computing a path-integral representation for the amplitude in quantum mechanics of having a transition from a quantum state $|q,t_0\rangle$ prepared as having a definite position q at time t_0 to a state with a different position at a different time $|\tilde{q},t_1\rangle$, where $t_1 > t_0$. We imagine the dynamics responsible for this evolution is described by a Hamiltonian $H(Q,P)$.

We work in the Heisenberg picture for which the quantum operators are time-dependent – $Q = Q(t)$ and $P = P(t)$ – with

$$Q(t) = e^{iHt}\, Q(0)\, e^{-iHt} \qquad \text{and} \qquad P(t) = e^{iHt}\, P(0)\, e^{-iHt}\,. \tag{7.5.3}$$

This implies a time dependence for their eigenvalues and eigenvectors as well,

$$Q(t)\,|q,t\rangle = q(t)\,|q,t\rangle \qquad \text{and} \qquad P(t)\,|p,t\rangle = p(t)\,|p,t\rangle\,, \tag{7.5.4}$$

where

$$|q,t\rangle = e^{iHt}\,|q,0\rangle \qquad \text{and} \qquad |p,t\rangle = e^{iHt}\,|p,0\rangle\,. \tag{7.5.5}$$

As mentioned in the previous section, the operators $Q(t)$ and $P(t)$ satisfy an equal-time commutation relation (7.5.1) that resembles the Poisson bracket relation[27] given in (7.2.13):

$$[Q(t)\,,P(t)] = i\,. \tag{7.5.6}$$

Standard arguments tell us that the commutation relations (7.5.6) imply the overlap beween position and momentum states is given at equal times by

$$\langle q,t|p,t\rangle = \frac{1}{\sqrt{2\pi}}\, e^{ipq}\,. \tag{7.5.7}$$

We seek the analogue of this expression for eigenstates evaluated at *different* times, using the Heisenberg-picture evolution (7.5.5) to do so. For infinitesimal time differences $t' = t + \mathrm{d}t$ one finds

$$\langle q,t+\mathrm{d}t|p,t\rangle = \langle q,t|\, e^{-iH(Q,P)\mathrm{d}t}\,|p,t\rangle \simeq e^{-iH(q,p)\mathrm{d}t}\langle q,t|p,t\rangle = \frac{1}{\sqrt{2\pi}}\, e^{-iH(q,p)\mathrm{d}t+ipq}\,, \tag{7.5.8}$$

where we assume $H(Q,P) = H[Q(t),P(t)]$ has the operator-ordering convention that all of the P's are written to the right of the Q's (as can always be ensured if not initially true by repeatedly using the commutation relation (7.5.6)).

[27] We work for convenience with units for which $\hbar = 1$, though the missing factors can easily be recovered at the end using dimensional analysis.

The expression for the overlap between two position eigenstates at slightly different times is then given by using the completeness of momentum eigenstates, which implies

$$\langle \psi | \phi \rangle = \int \mathrm{d}p \; \langle \psi | p \,, t \rangle \, \langle p \,, t | \phi \rangle \,, \tag{7.5.9}$$

for *any* two states in the quantum Hilbert space. This is called 'inserting a partition of unity'. Doing so in the amplitude $\langle q, t + \mathrm{d}t | \tilde{q}, t \rangle$ and using (7.5.7) and (7.5.8) gives:

$$\langle q, t + \mathrm{d}t | \tilde{q}, t \rangle = \int \mathrm{d}p \; \langle q, t + \mathrm{d}t | p \,, t \rangle \, \langle p \,, t | \tilde{q}, t \rangle = \int \frac{\mathrm{d}p}{\sqrt{2\pi}} \; e^{-iH(q,p)\mathrm{d}t + ip(q - \tilde{q})} \,. \tag{7.5.10}$$

Now comes the main point. An expression for the overlap between position eigenstates separated by a finite time difference $T := t_f - t_i > 0$ can be found by inserting a partition of unity using the complete position eigenstates at slightly different times, each displaced relative to the previous one by a time step $\mathrm{d}\tau$. That is, define $\mathcal{N} + 2$ evenly spaced instants in time, t_k, chosen such that $t_0 = t_i$ and $t_{\mathcal{N}+1} = t_f$ and with $\Delta t := t_{k+1} - t_k = T/(\mathcal{N} + 1)$ approaching $\mathrm{d}t$ as $\mathcal{N} \to \infty$. Then

$$\langle q_f, t_f | q_i, t_i \rangle = \int \mathrm{d}q_1 \cdots \mathrm{d}q_{\mathcal{N}} \; \langle q_f, t_f | q_{\mathcal{N}}, t_{N} \rangle \cdots \langle q_2, t_2 | q_1, t_1 \rangle \, \langle q_1, t_1 | q_i, t_i \rangle \,, \tag{7.5.11}$$

and so using (7.5.10) for each of these overlaps gives

$$\langle q_f, t_f | q_i, t_i \rangle = \int \prod_{k=1}^{\mathcal{N}} \frac{\mathrm{d}q_j}{\sqrt{2\pi}} \; \prod_{j=0}^{\mathcal{N}} \frac{\mathrm{d}p_j}{\sqrt{2\pi}} \; \exp\left\{ i \sum_{l=1}^{\mathcal{N}+1} \Big[(p_{l-1})(q_l - q_{l-1}) - H(q_l, p_{l-1}) \, \Delta t \Big] \right\} \,, \tag{7.5.12}$$

where $q_0 := q_i$ and $q_{\mathcal{N}+1} := q_f$ are not integrated.

This becomes a path integral in the limit $\mathcal{N} \to \infty$. To see why, write the sequence of values $\{q_0, t_0; q_1, t_1; \cdots; q_{\mathcal{N}}, t_{\mathcal{N}}; q_{\mathcal{N}+1}, t_{\mathcal{N}+1}\}$ and $\{p_0, t_0; p_1, t_1; \cdots; p_{\mathcal{N}}, t_{\mathcal{N}}; p_{\mathcal{N}+1}, t_{\mathcal{N}+1}\}$ as curves $q(t)$ and $p(t)$ sampled at each of the t_l's with the endpoints of $q(t)$ anchored at the initial and final positions:

$$q(t_i) = q_i \qquad \text{and} \qquad q(t_f) = q_f \,. \tag{7.5.13}$$

The sum in the exponent of (7.5.12) becomes an integral involving these curves as $\mathcal{N} \to \infty$, with $\Delta t \to \mathrm{d}t$ and $q_l - q_{l-1} \to \dot{q}(t)\,\mathrm{d}t$ in this limit, leading to the main result

$$\langle q_f, t_f | q_i, t_i \rangle = \int_{q_i}^{q_f} \mathcal{D}q(t) \int \mathcal{D}p(t) \; \exp\left\{ i \int_{t_i}^{t_f} \mathrm{d}t \Big[p(t)\, \dot{q}(t) - H[q(t), p(t)] \Big] \right\} \,. \tag{7.5.14}$$

The notation $\int \mathcal{D}q(t)$ and $\int \mathcal{D}p(t)$ denotes *functional* integration, meaning an integration over *all* possible different choices for the different paths $q(t)$ and $p(t)$. All curves $q(t)$ and $p(t)$ are included in this integral – not just $q(t)$ and $p(t)$ satisfying the classical equations of motion – and each curve contributes a phase to the overall amplitude given by the integrand of (7.5.14). The limits on the $\mathcal{D}q(t)$ integration in (7.5.14) mean that only curves satisfying (7.5.13) are to be summed over, while the absence of limits on the $\int \mathcal{D}p(t)$ integral means this includes an integration over the initial and final values for $p(t_i)$ and $p(t_f)$.

Comparing with expressions like (7.1.5) shows that the phase weighting each path is controlled by the action. Replacing the factors of $\hbar$ on dimensional grounds then shows that each path contributes the amplitude

$$e^{iS[q(t),p(t)]/\hbar} . \tag{7.5.15}$$

The total amplitude is then obtained by interfering all of these amplitudes together by summing the contribution over all possible paths. This is a very profound reframing of what quantum evolution really is.

In particular it provides insight into why the classical path, obtained by finding the stationary point of the action, is important. In classical systems the value of the action is much much larger than the value of $\hbar$ and so the phase of the amplitude $e^{iS/\hbar}$ changes very rapidly as one varies through different paths. This generically leads to destructive interference, with the contributions of neighbouring paths cancelling one another out. But the same is *not* true for paths near a stationary point of the action, however, since for these paths the amplitudes interfere constructively because the action (by assumption) does not change by much.

It is this constructive interference of neighbouring paths that ensures that classical paths dominate within quantum amplitudes, at least in the classical limit where $S \gg \hbar$.

8 Chaotic Evolution

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8.1 Integrable Systems*

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8.1.1 Action-Angle variables

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8.1.2 Liouville's Theorem

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8.2 Dynamical Systems*

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8.2.1 Phase Portraits

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8.3 Chaotic evolution*

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8.3.1 Adiabatic Invariants

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9 Continuum Mechanics

This section picks up where §5.3 left off: a description of the macroscopic motion whose roots lie in the motion of more microscopic constituents. From this point of view it is a continuation of the theme of providing ever-finer detail about the motion of macroscopic bodies that started with centre-of-mass motion, proceeded through rigid-body motion to the pattern of small oscillations of atoms near their equilibrium positions.

This section extends the earlier treatment by making it more systematic. This also allows a discussion that goes beyond the simplest systems to include shear and other types of macroscopic manifestations of interatomic restoring forces. It also sets up the generalization – given in §9.3 – of Newton's laws to describe the macroscopic motion of *fluids*: systems for which the constituent atoms do not have unique equilibrium positions relative to one another.

Together these two topics (elasticity and fluid mechanics) form the core of the applications of Newton's Laws to the more detailed behaviour of macroscopic collections of atoms in the regime where spatial resolution is insufficient to resolve the atoms and so justifies the system's approximate treatment as a continuous medium.

For a continuous system we imagine that there is a smallest volume, $\mathrm{d}V$, that we can resolve. This volume is on one hand assumed to be much smaller than is the physical system of interest and on the other hand is also assumed to be much larger than the typical spacing between the underlying constituent atoms. Rather than applying Newton's laws atom by atom we instead ask how they apply to such a minimal volume element. To this end this section describes how applied forces are represented within the continuum limit.

When doing so it is worth keeping in mind that the underlying atomic picture tells us that extensive properties like total mass or total particle number of the volume element will be explicitly proportional to its volume $\mathrm{d}V = \mathrm{d}^3x = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, so

$$\mathrm{d}m = \rho\,\mathrm{d}V \quad \text{and} \quad \mathrm{d}N = n\,\mathrm{d}V\,, \tag{9.0.1}$$

where coefficients like $\rho(\mathbf{r},t) = \mathrm{d}m/\mathrm{d}V$ or $n(\mathbf{r},t) = \mathrm{d}N/\mathrm{d}V$ depend on the local properties of the underlying atoms but not on the size of the volume element itself.

9.1 Stress

The goal is to implement Newton's laws by identifying the momentum of the volume element and equating its time rate of change to the net force acting on the volume element. Our guideline when doing so is to have the macroscopic expressions parallel the properties found back in §1.3 when discussing the implications of Newton's laws for macroscopic objects atom by atom. This comparison to the atomic point of view suggests there are two types of forces to keep track of: body forces like the force of gravity or electromagnetism that act at a distance, and the inter-atomic forces that, when short-ranged, cause adjacent volume elements to act on one another.

9.1.1 Long-range body forces

The force of gravity acting on each volume element is simple to write:

$$\mathrm{d}\mathbf{F}_g = \mathbf{g}\,\mathrm{d}m = \rho\,\mathbf{g}\,\mathrm{d}V\,, \tag{9.1.1}$$

where the mass density $\rho(\mathbf{r},t)$ is evaluated at the position of the volume element of interest and $\mathbf{g}(\mathbf{r},t)$ can be imagined to differ for different volume elements. The gravitational force acting on a larger region $\mathcal{R}$ is then found simply by adding the contribution from each of its volume elements

$$\mathbf{F}_g(\mathcal{R}) = \int_{\mathcal{R}} \mathrm{d}\mathbf{F}_g = \int_{\mathcal{R}} \rho\,\mathbf{g}\,\mathrm{d}V\,. \tag{9.1.2}$$

It is often useful to rewrite the gravitational body force in terms of the gravitational potential Φ_g, in terms of which $\mathbf{g}(\mathbf{r},t) = -\nabla\Phi_g(\mathbf{r},t)$.

Electrostatic forces can be written in a similar way. If $\mathrm{d}q = \mathfrak{q}\,\mathrm{d}V$ is the electric charge of the volume element, with $\mathfrak{q}(\mathbf{r},t) = \mathrm{d}q/\mathrm{d}V$ the electric charge density, then the electric force on the element due to an applied electric field $\mathbf{E}$ is

$$\mathrm{d}\mathbf{F}_E = \mathrm{d}q\,\mathbf{E} = \mathfrak{q}\mathbf{E}\,\mathrm{d}V \tag{9.1.3}$$

and so the electric force on a region $\mathcal{R}$ containing many such volume elements is

$$\mathbf{F}_E(\mathcal{R}) = \int_{\mathcal{R}} \mathrm{d}\mathbf{F}_E = \int_{\mathcal{R}} \mathfrak{q}\mathbf{E}\,\mathrm{d}V = \int_{\mathcal{R}} \gamma\rho\,\mathbf{E}\,\mathrm{d}V\,, \tag{9.1.4}$$

where the last equality introduces the local charge-to-mass ratio: $\gamma(\mathbf{r},t) = \mathfrak{q}(\mathbf{r},t)/\rho(\mathbf{r},t)$.

9.1.2 Short-range interatomic forces

Writing down the macroscopic version of forces acting between different volume elements is slightly trickier. This can be done relatively simply, however, when these forces act only over atomic distances, which are by assumption much smaller than the size of the volume elements of interest.

As always, the net force on the volume element is found by summing over the forces acting on all of the atoms within the volume element. But we also know that Newton's third law ensures that the forces between pairs of atoms that are both interior to the same volume element cancel out when this is done, along the lines found when deriving (1.3.2) from (1.3.1). So the only forces that matter for any one volume element must be applied by atoms that are not in the volume element in question. But if these forces are all very short ranged then these atoms must be very close by and the force that they apply can be regarded as being applied to the surface of the volume element.

Importantly, the same must also be true for the forces applied to larger regions built from many smaller volume elements. If $\mathcal{R}$ consists of the union of a number of volume elements then the net force acting on $\mathcal{R}$ due to short-range interatomic forces must be expressible in two separate ways:

1. It must be given as a sum over the forces applied to each of its constituent volume elements.

2. The result must also be expressable as a sum of forces only applied to its boundary surface $\partial\mathcal{R}$ *regardless of the shape of* $\mathcal{R}$.

Now comes the main point: suppose the components, $\mathrm{d}F^i$, of the interatomic forces acting on any particular volume element are given by

$$\mathrm{d}F^i = \mathcal{F}^i \mathrm{d}V\,. \tag{9.1.5}$$

Then the above two conditions can only be consistent if $\mathcal{F}^i$ is the derivative of some quantity:

$$\mathcal{F}^i = \frac{\partial \tau^{ij}}{\partial x^j} = \partial_j \tau^{ij} \,, \tag{9.1.6}$$

because when this is so then the components of the net force acting on a larger region $\mathcal{R}$ becomes

$$F^i(\mathcal{R}) = \int_{\mathcal{R}} \frac{\partial \tau^{ij}}{\partial x^j} \, \mathrm{d}V = \oint_{\partial\mathcal{R}} n_j \, \tau^{ij} \, \mathrm{d}S \,. \tag{9.1.7}$$

Here the second equality evaluates the volume integral using Gauss' theorem to rewrite the integral over a total derivative as the integral over the boundary of $\mathcal{R}$. Here $\mathrm{d}S$ is a surface area element and $\mathbf{n}$ is the outward-pointing unit normal at the surface element $\mathrm{d}S$.

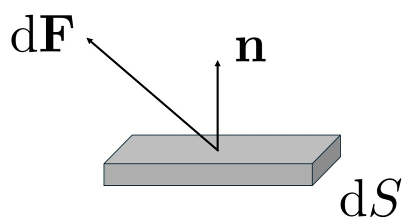

Figure 28. Sketch of the force $\mathrm{d}\mathbf{F}$ applied to a small surface element of a region $\mathcal{R}$, together with the surface element's outward-pointing unit normal $\mathbf{n}$. The force components and the components of $\mathbf{n}$ are related by $\mathrm{d}F^i = n_j \, \tau^{ij}$ where τ^{ij} is the local stress tensor.

The quantity τ^{ij} is called the *stress tensor*. The final equality of (9.1.7) shows that it gives the direction of the force acting on any surface element relative to the surface's normal. Diagonal elements like τ^{xx}, τ^{yy} and τ^{zz} describe forces that are normal to the surface in question (as would be the force due to a pressure within the material) while off-diagonal components like τ^{xy}, τ^{xz}, τ^{yz} and so on describe forces that act tangent to the surface.

The torque, $\mathbf{r} \times \mathrm{d}\mathbf{F}$ due to the net force on any surface element has components $\mathrm{d}\mathfrak{t}_i = \epsilon_{ijk} x^j \mathrm{d}F^k$ and so is sensitive to the antisymmetric product $x^j \mathrm{d}F^k - x^k \mathrm{d}F^j$. Using (9.1.7) this can be written

$$\Big[x^i \partial_k \tau^{jk} - x^j \partial_k \tau^{ik}\Big] \mathrm{d}V = \Big[\partial_k \Big(x^i \tau^{jk} - x^j \tau^{ik}\Big) - \tau^{ji} + \tau^{ij}\Big] \mathrm{d}V \,. \tag{9.1.8}$$

Integrating over a region $\mathcal{R}$ within a medium then gives

$$\int_{\mathcal{R}} \Big[x^i \partial_k \tau^{jk} - x^j \partial_k \tau^{ik}\Big] \mathrm{d}V = \oint_{\partial\mathcal{R}} \mathrm{d}S \, n_k \Big(x^i \tau^{jk} - x^j \tau^{ik}\Big) + \int_{\mathcal{R}} \mathrm{d}V \, \Big(\tau^{ij} - \tau^{ji}\Big) \,. \tag{9.1.9}$$

This should also be writable as a surface term because of the short range of the interatomic forces and this shows that the stress tensor must be symmetric:

$$\tau^{ij} = \tau^{ji} \,. \tag{9.1.10}$$

Because of this the torque on any region $\mathcal{R}$ within the medium can be written

$$\mathfrak{t}_i(\mathcal{R}) = \tfrac{1}{2} \oint_{\partial \mathcal{R}} \mathrm{d}S \, \epsilon_{ijk} n_l \Big(x^j \tau^{kl} - x^k \tau^{jl} \Big) \,. \tag{9.1.11}$$

In the simplest case where the strain tensor is spherically symmetric it must be proportional to the identity matrix, since this is the only rotationally invariant symmetric tensor (see §A.3.1). So in this case

$$\tau^{ij}_{\rm sym} = -p \, \delta^{ij} \,, \tag{9.1.12}$$

for some quantity p. The force acting on a surface element is then in the direction of the surface's outward pointing unit normal,

$$\mathrm{d}F^i = \tau^{ij}_{\rm sym} \, n_j \, \mathrm{d}S = -p \, n^i \, \mathrm{d}S \quad \text{or} \quad \mathrm{d}\mathbf{F} = -p \, \mathbf{n} \, \mathrm{d}S \,, \tag{9.1.13}$$

showing that $p = \mathrm{d}F/\mathrm{d}S$ is the magnitude of the *inward-directed* applied force per unit area and so is naturally interpreted as the medium's pressure.[28]

Given this characterization of how forces act on different volumes within a medium we are now in a position to express the implications of Newton's 2nd law for continuous media. We first identify the momentum of each volume element $\mathrm{d}\mathbf{p} = \boldsymbol{\pi} \, \mathrm{d}V$ and then equate its temporal rate of change to the net applied force, as computed above, leading to

$$\partial_t \pi^i = \partial_k \tau^{ik} + (F_B)^i \,, \tag{9.1.14}$$

where $\mathbf{F}_B$ denotes the net body force applied.

Once this is done the momentum density $\boldsymbol{\pi}$ and the stress tensor must be expressed in terms of the medium's motion in order to get a differential equation that can be solved for this motion. How this is done depends on whether or not the constituent atoms have equilibrium positions – *i.e. elastic* media – or are free to move relative to one another – *i.e. fluids* – so each of these is considered in turn in the next two sections.

9.2 Elastic Media

When atoms have fixed equilibrium positions relative to their neighbours then the dynamical variable is the local displacement of the volume element from its equilibrium position. This can be described within a continuous medium by a displacement vector $\mathbf{u}(\mathbf{r}, t)$, with the volume element's new position $\tilde{\mathbf{r}}$ and old position related by

$$\tilde{\mathbf{r}} = \mathbf{r} + \mathbf{u}(\mathbf{r}, t) \,. \tag{9.2.1}$$

[28]Recall that τ^{ij} captures the force acting on the surface of a region $\mathcal{R}$ coming from the surrounding media, and so should be inward directed if this force arises because the medium has an ambient pressure (this is the reason for the negative sign in the definition (9.1.12)).

Our assumed inability to resolve distances smaller than the size of the volume element means we can safely restrict attention to configurations for which all atoms within a single volume element share the same displacement.

With this definition the goal now is to express both the momentum of a volume element and the applied forces in terms of the dynamical variable $\mathbf{u}(\mathbf{r}, t)$. Of these, the momentum is easiest to do, since for a specific volume element the momentum is the sum of the momenta of the individual atoms and so

$$\mathrm{d}\mathbf{p} = \mathrm{d}m\, \partial_t \mathbf{u} = \rho\, \partial_t \mathbf{u}\, \mathrm{d}V \quad \text{and so} \quad \boldsymbol{\pi} = \rho\, \partial_t \mathbf{u}\,. \tag{9.2.2}$$

This implies the time rate-of-change of the momentum in a fixed region $\mathcal{R}$ whose position and boundaries do not move is given by

$$\partial_t \mathbf{p}(\mathcal{R}) = \partial_t \int_{\mathcal{R}} \mathrm{d}\mathbf{p} = \int_{\mathcal{R}} \mathrm{d}V\; \partial_t\Big(\rho\, \partial_t \mathbf{u}\Big)\,. \tag{9.2.3}$$

The local expression of Newton's second law for elastic materials is then obtained everywhere within the continuous medium by combining this with eq. (9.1.14) to write

$$\partial_t\Big(\rho\, \partial_t u^i\Big) = \partial_k \tau^{ik} + (F_B)_i\,, \tag{9.2.4}$$

where $\mathbf{F}_B$ denotes the net body force. In order to use this expression we need to determine how τ^{ik} depends on the displacement $\mathbf{u}$.

9.2.1 The Strain Tensor

For elastic materials it is displacement away from the equilibrium position that produces the forces that push the atoms back to where they belong. This means that it should be possible to express the stress tensor τ^{ij} in terms of the displacement $\mathbf{u}$ (at least for small $\mathbf{u}$). To the extent that the strength of interatomic forces is a function of atomic separations this dependence should enter at a macroscopic level through the dependence of the forces on the physical distances between volume elements.

Consider, then, two volume elements initially separated by a given coordinate distance $\mathrm{d}\mathbf{r} = \mathrm{d}x\, \mathbf{e}_x + \mathrm{d}y\, \mathbf{e}_y + \mathrm{d}z\, \mathbf{e}_z$. The physical distance between these two volume elements then is $\mathrm{d}s = \sqrt{\mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r}}$. After these volume elements are displaced by the displacement field $\mathbf{u}(\mathbf{r}, t)$ the distance between them instead becomes $\mathrm{d}\tilde{s} = \sqrt{\mathrm{d}\tilde{\mathbf{r}} \cdot \mathrm{d}\tilde{\mathbf{r}}}$ where the differential version of (9.2.1) implies $\mathrm{d}\tilde{\mathbf{r}} = \mathrm{d}\mathbf{r} + (\mathrm{d}\mathbf{r} \cdot \nabla)\mathbf{u}$. In component form this becomes $\mathrm{d}\tilde{x}^i = \mathrm{d}x^i + \mathrm{d}x^j \partial_j u^i$.

The distance therefore satisfies

$$\mathrm{d}\tilde{s}^2 = \delta_{ij}(\mathrm{d}x^i + \mathrm{d}x^k \partial_k u^i)(\mathrm{d}x^j + \mathrm{d}x^l \partial_l u^j) = \mathrm{d}s^2 + 2\, \sigma_{ij}\, \mathrm{d}x^i \mathrm{d}x^j\,, \tag{9.2.5}$$

where the *strain tensor* σ_{ij} is defined by

$$\sigma_{ij} = \tfrac{1}{2}\left[\frac{\partial u_j}{\partial x^i} + \frac{\partial u_i}{\partial x^j} + \frac{\partial u^k}{\partial x^i}\,\frac{\partial u_k}{\partial x^j}\right]. \tag{9.2.6}$$

The strain tensor is by definition symmetric, $\sigma_{ij} = \sigma_{ji}$, since any antisymmetric contribution would not contribute to (9.2.5). In practice our interest is only in situations for which atoms are displaced through distances much smaller than the distances we can resolve, in which case the components $|\partial_i u_j|$ are small and it suffices to use

$$\sigma_{ij} \simeq \tfrac{1}{2}\Big(\partial_i u_j + \partial_j u_i\Big) \,. \tag{9.2.7}$$

Notice that the Jacobian of the transformation from x^i to $\tilde{x}^i = x^i + u^i$ is $\mathcal{J}^i_j = \partial \tilde{x}^i / \partial x^j = \delta^i_j + \partial_j u^i$, and so the element of volume $\mathrm{d}^3x = \mathrm{d}V$ itself changes by

$$\mathrm{d}V \to \mathrm{d}\widetilde{V} = \mathrm{d}V\,|\det \mathcal{J}| \simeq \mathrm{d}V\Big(1 + \partial_i u^i\Big) = \mathrm{d}V\Big(1 + \nabla \cdot \mathbf{u}\Big) \,, \tag{9.2.8}$$

where $|\det \mathcal{J}|$ denotes the absolute value of the determinant of the Jacobian. The approximate equality evaluates this by applying the identity

$$\det A = \exp\Big[\mathrm{tr}\Big(\log A\Big)\Big] \tag{9.2.9}$$

(which holds for any diagonalizable matrix A) and linearizing the result in the small components $\partial_i u^j$. Eq. (9.2.9) can be derived from the expressions $\det A = \prod_i \lambda_i$ and $\mathrm{tr}\, A = \sum_i \lambda_i$ where λ_i are the eigenvalues of A. Specializing (9.2.9) to $A = A_0 + \delta A$ and linearizing in δA implies $\det A = \det A_0 + \delta(\det A)$ where

$$\delta(\det A) = \delta[\mathrm{tr}(\log A)] \det A_0 = [\mathrm{tr}(A_0^{-1}\, \delta A)] \det A_0 \,, \tag{9.2.10}$$

and the linearization in eq. (9.2.8) uses the special case $A_0 = I$ and $\mathrm{tr}(\delta A) = \nabla \cdot \mathbf{u}$.

The physical significance of (9.2.8) is that the divergence of $\mathbf{u}$ expresses the fractional change of volume that is induced by the displacement field $\mathbf{u}$:

$$\frac{\mathrm{d}\widetilde{V} - \mathrm{d}V}{\mathrm{d}V} = \nabla \cdot \mathbf{u} \,. \tag{9.2.11}$$

Notice that this means the fractional change of volume induced by the displacement $\mathbf{u}$ is given (for small $\partial_i u^j$) by the trace of the strain tensor (9.2.7), because

$$\mathrm{tr}\, \sigma = \delta^{ij} \sigma_{ij} \simeq \partial_i u^i = \nabla \cdot \mathbf{u} \,. \tag{9.2.12}$$

In what follows it can be convenient to separate the strain tensor (9.2.7) into an 'expansion' part proportional to the unit matrix and a 'shear' part that is trace-free: $\sigma_{ij} = \sigma^{\mathrm{sh}}_{ij} + \sigma^{\mathrm{ex}}_{ij}$ with

$$\sigma^{\mathrm{sh}}_{ij} := \tfrac{1}{2}\Big[\partial_i u_j + \partial_j u_i - \tfrac{2}{3}\,(\nabla \cdot \mathbf{u})\, \delta_{ij}\Big] \quad \text{and} \quad \sigma^{\mathrm{ex}}_{ij} := \tfrac{1}{3}(\nabla \cdot \mathbf{u})\, \delta_{ij} \,. \tag{9.2.13}$$

9.2.2 Energy and local equilibrium

This section records the energy associated with the deformation of an elastic material, keeping in mind when doing so that any unseen random motions of the underlying atoms also carry energy.

The starting point is the work done by a small virtual change $\delta\mathbf{u}$ in the atomic displacement. This is computed for each volume element by (as usual) taking the inner product of $\delta\mathbf{u}$ with the applied force, so $\delta W = \delta\mathbf{u}\cdot \mathrm{d}\mathbf{F} = \delta u_i\, \mathcal{F}^i\, \mathrm{d}V = \delta u_i\,(\partial_j \tau^{ij})\,\mathrm{d}V$. Summing this up over a region $\mathcal{R}$ in the medium then gives

$$\begin{aligned} \delta W(\mathcal{R}) &= \int_{\mathcal{R}} \delta u_i\,(\partial_j\tau^{ij})\,\mathrm{d}V = \oint_{\partial\mathcal{R}} \delta u_i\,\tau^{ij}\,n_j\,\mathrm{d}S - \tfrac{1}{2}\int_{\mathcal{R}} \tau^{ij}\,(\partial_j\delta u_i + \partial_i \delta u_j)\,\mathrm{d}V \\ &= \oint_{\partial\mathcal{R}} \delta u_i\,\tau^{ij}\,n_j\,\mathrm{d}S - \int_{\mathcal{R}} \tau^{ij}\,\delta\sigma_{ij}\,\mathrm{d}V\,, \end{aligned} \tag{9.2.14}$$

where the last equality on the first line integrates by parts, and uses the symmetry property $\tau^{ij} = \tau^{ji}$ to write the derivative of $\delta\mathbf{u}$ in a symmetric way. The final line uses the definition (9.2.7) for the strain to identify how it varies when $\mathbf{u}$ varies.

The last term in (9.2.14) takes a more familiar form if specialized to a rotationally invariant medium for which $\tau^{ij} = -p\,\delta^{ij}$, where p is the medium's ambient pressure (see the discussion surrounding eq. (9.1.13)). In this case

$$-\tau^{ij}\delta\sigma_{ij}\mathrm{d}V = p\,\delta\,(\nabla\cdot\mathbf{u})\,\mathrm{d}V = p\,\delta(\mathrm{d}V)\,, \tag{9.2.15}$$

where the first equality uses (9.2.12) and the final equality uses (9.2.11). The last term is a contribution to the work done by a system's pressure when the volume is changed, as is commonly encountered in elementary courses on thermodynamics. The connection to thermodynamics is not accidental because from a macroscopic point of view it is impossible to be sure that any applied energy goes entirely into coherently displacing atoms within a volume element as opposed to giving them incoherent random motions relative to one another.

The energy associated with this kind of random microscopic motion can be handled thermodynamically however if the deformation $\delta\mathbf{u}(\mathbf{r},t)$ takes place slowly enough to allow the system to remain in local thermal equilibrium at all times. Here 'local equilibrium' means that any thermodynamic variables (like internal energy, particle number, pressure, temperature *etc*) are related as described in introductory thermodynamics classes, but with all of these variables (which are imagined to be functions of position and time) evaluated at the same place.

For instance the ideal gas law says[29] $pV = NT$ (in fundamental units, for which Boltzmann's constant is $k_B = 1$), where p is the pressure, V is the system volume, N is the number

[29] An ideal gas is used here despite ideal gasses being more fluids than elastic media because the ideal gas law is a particularly well-known equation of state. (See §9.3 for more about fluids.) The broader point made here about local equilibrium applies equally well to equations of state more appropriate to elastic media.

of gas particles and T is the temperature. An equivalent way to write this is $p = n\,T$ where $n = N/V$ is the number density of particles, which is more useful for continuous media because n (unlike N and V) can be regarded as varying in space and time (just like p and T). Local equilibrium means $p(\mathbf{r},t) = n(\mathbf{r},t)\,T(\mathbf{r},t)$ locally holds for all positions and times. The local value of any thermodynamic quantity can be regarded as being the approximately constant value it takes across a small volume element $\mathrm{d}V$.

It might seem puzzling to work with a local equation of state because in the usual story of statistical mechanics thermodynamics emerges in the limit $N, V \to \infty$ with N/V fixed, in which case statistical fluctuations go to zero. How can infinite volume be consistent with a picture where thermodynamic variables arise from the statistical mechanics of the contents of very small volume elements? The key idea here is separation of scales: the continuum limit assumes $\ell \ll a$: *i.e.* microscopic inter-atomic scales, ℓ, are much smaller than the small size, a, of the volume elements resolved in a continuum description. It also assumes a itself is much smaller than the sizes of objects being described in the continuum limit (like the wavelength, λ, of a propagating wave).

Since particles and energy can both flow between adjacent volume elements as atoms move, the appropriate statistical ensemble for each volume element is the grand canonical ensemble (for which energy and particle number are not fixed). In this ensemble both the energy $\mathrm{d}U = u\,\mathrm{d}V$ and the particle number $\mathrm{d}N = n\,\mathrm{d}V$ in a volume element fluctuate between different representatives of the statistical ensemble. But in a volume of size a^3 fluctuations like Δn or Δu are suppressed compared to their mean values $\overline{n}$ and $\overline{u}$ by powers of ℓ/a (which is why they go to zero in the infinite-volume limit). For finite a these fluctuations are nonzero but we assume ℓ is small enough relative to a that the fractional size of these thermal fluctuations is negligible.

Local equilibrium in continuum mechanics further assumes that a/λ is small enough that *e.g.* the change $\delta\overline{u} \sim a\,\partial\overline{u}$ in $\overline{u}$ between adjacent volume elements (that are separated by a distance that is order a in size) is smaller than the thermal fluctuations in either of the volume elements separately: $\delta\overline{u} < \Delta u$. It is because δu is smaller than Δu that we can treat $\overline{u}$ as being approximately constant within any given volume element while still imagining the average $\overline{u}$ can drift over macroscopic distances. Both $\Delta u \ll \overline{u}$ and $\delta\overline{u} < \Delta u$ can be satisfied if ℓ is sufficiently small compared with a and a is sufficiently small compared to λ (*i.e.* if derivatives like $\partial\overline{u}$ are not too big).

Returning to the main story, when local equilibrium applies the contribution of energy transfer into random atomic motions is encorporated by including a change to the internal energy, $U = \int u\,\mathrm{d}V$, due to a local transfer of heat. In equilibrium heat transfer can be written as a contribution $\int T\,\delta s\,\mathrm{d}V$ where $T(\mathbf{r},t)$ is the medium's local temperature and $\delta s(\mathbf{r},t)$ is the local change to the medium's entropy per unit volume $\mathcal{S} = \int s\,\mathrm{d}V$.

Combining both the work done by strain and the energy change due to heating, the change to the internal energy in a given volume element is $\delta u = T\,\delta s - \delta W$ and so the change in the total energy is

$$\delta U = \int \delta u \,\mathrm{d}V = \int \Big(T\delta s + \tau^{ij}\,\delta\sigma_{ij}\Big)\mathrm{d}V \tag{9.2.16}$$

where the integral is now over the entire volume of the medium and the surface term in the work done – *c.f.* eq. (9.2.14) – is dropped because the displacement $\delta\mathbf{u}$ is taken to vanish at the medium's physical boundary.

Equating integrands shows that the internal energy has as its natural variables the entropy and the system strain: $u = u(s, \sigma_{ij})$, and

$$\left(\frac{\partial u}{\partial s}\right)_\sigma = T \quad \text{and} \quad \left(\frac{\partial u}{\partial \sigma_{ij}}\right)_s = \tau^{ij}\,, \tag{9.2.17}$$

where the subscripts emphasize what is being held fixed when the derivatives are taken. This is the generalization of the usual statement $u = u(s, V)$ and $\mathrm{d}u = T\mathrm{d}s - p\,\mathrm{d}V$ for more elementary systems for which the only change in shape explored is a change to the overall volume.

It is often more useful to have T as the independent variable rather than the entropy density s and to this end we define the Helmholtz energy in the usual way: $F = U - T\mathcal{S}$. Writing $F = \int f\,\mathrm{d}V$ we find $\delta f = -s\,\delta T + \tau^{ij}\,\delta\sigma_{ij}$ and so

$$\left(\frac{\partial f}{\partial T}\right)_\sigma = -s \quad \text{and} \quad \left(\frac{\partial f}{\partial \sigma_{ij}}\right)_T = \tau^{ij}\,. \tag{9.2.18}$$

A Gibbs energy $G = F - \int \tau^{ij}\sigma_{ij}\mathrm{d}V$ similarly provides a thermodynamic potential whose natural variables are temperature T and stress τ^{ij}, in which case $G = \int g\,\mathrm{d}V$ implies

$$\left(\frac{\partial g}{\partial T}\right)_\tau = -s \quad \text{and} \quad \left(\frac{\partial g}{\partial \tau^{ij}}\right)_T = -\sigma_{ij}\,. \tag{9.2.19}$$

9.2.3 Constitutive Relations: Hooke's law

With the definition of the strain tensor under our belts together with the above way to characterize the energy associated with changes to strain we can return to the problem of determining the expression for the stress in terms of the atomic displacements $\mathbf{u}$. We do so here by providing general expressions for the energy as a function of strain and then differentiating to get the stress using (9.2.18).

For small strains the simplest procedure is to write the free energy F as a Taylor expansion in powers of σ_{ij}. The leading term arises at quadratic order because the configuration $\mathbf{u} = 0$ has been assumed to be an equilibrium configuration for which the stress $\tau^{ij} = \delta F/\delta\sigma_{ij}$ vanishes.[30] This suggests the leading small-strain form for $F = \int f\,\mathrm{d}V$ has f written as a quadratic function of σ_{ij}:

$$f = f_0 + \tfrac{1}{2}\, c^{ijkl}\sigma_{ij}\sigma_{kl}\,, \tag{9.2.20}$$

[30] It can happen that the equilibrium position depends on temperature, in which case the free energy density can contain a term linear in the strain, like $\alpha\,(\mathrm{tr}\,\sigma)$ where α vanishes for a specific temperature $T = T_0$.

for some medium-dependent coefficients c^{ijkl} that satisfy $c^{ijkl} = c^{klij} = c^{ijlk} = c^{jikl}$. For rotationally invariant systems the coefficients c^{ijkl} must be built using only the Kronecker delta and there are only two independent ways to do so consistent with the symmetries: $c^{ijkl} \propto \delta^{ij}\delta^{kl}$ and $c^{ijkl} \propto \delta^{ik}\delta^{jl} + \delta^{jk}\delta^{il}$. Their coefficients can be chosen such that

$$f = f_0 + \tfrac{1}{2}\,\lambda(\mathrm{tr}\,\sigma)^2 + \mu\,\sigma_{ij}\sigma^{ij}\,, \tag{9.2.21}$$

where λ and μ are called the medium's *Lamé coefficients*. The signs in this expression are chosen so that nonzero strain increases the energy when λ and μ are positive.

A common alternative parameterization instead splits the strain into its shear and expansion parts – *c.f.* eq. (9.2.13) – and so replaces (9.2.21) with

$$f = f_0 + \mu\left(\sigma_{ij} - \tfrac{1}{3}\,\delta_{ij}\,\mathrm{tr}\,\sigma\right)^2 + \tfrac{1}{2}\kappa(\mathrm{tr}\,\sigma)^2\,, \tag{9.2.22}$$

and so

$$\kappa = \lambda + \tfrac{2}{3}\mu\,. \tag{9.2.23}$$

In this way of parameterizing things κ is called the *bulk modulus* or the *compression modulus* while μ is called the *shear modulus*. This second parameterization has the advantage that the shear and expansion contributions to the strain are independent of one another and so the condition that the equilibrium configuration $\mathbf{u} = 0$ is stable is $\kappa > 0$ and $\mu > 0$.

The stress induced by a given strain can now be found by differentiating (9.2.22) and using the result in (9.2.18), leading to

$$\tau_{ij} = \frac{\partial f}{\partial \sigma_{ij}} = 2\mu\Big[\sigma_{ij} - \tfrac{1}{3}\,\delta_{ij}\,(\mathrm{tr}\,\sigma)\Big] + \kappa\,\delta_{ij}\,(\mathrm{tr}\,\sigma) = 2\mu\,\sigma_{ij} + \lambda\,\delta_{ij}\,(\mathrm{tr}\,\sigma)\,. \tag{9.2.24}$$

For suffiiciently small strains the stress is linear in the strain. This is the general formulation of Hooke's law for elastic media.

It is sometimes desirable to compute the strain that results when a given stress is applied and this involves inverting (9.2.24) to solve for σ_{ij} as a function of τ_{ij}. The first step in this direction takes the trace of (9.2.24) to learn

$$\mathrm{tr}\,\tau = 3\kappa\,\mathrm{tr}\,\sigma = 3\kappa(\nabla\cdot\mathbf{u})\,. \tag{9.2.25}$$

Using this to eliminate $\mathrm{tr}\,\sigma = \nabla\cdot\mathbf{u}$ from (9.2.24) leaves a result that can be solved for σ_{ij}, leading to

$$\sigma_{ij} = \frac{1}{2\mu}\Big[\tau_{ij} - \tfrac{1}{3}\,\delta_{ij}\,(\mathrm{tr}\,\tau)\Big] + \frac{\mathrm{tr}\,\tau}{9\kappa}\,\delta_{ij}\,. \tag{9.2.26}$$

Worked example: Strain produced by a known stress

Consider an initially cubic elastic material with given bulk and shear moduli, κ and μ. Suppose the stress within the material is constant and that the faces of the cube are parallel to the x, y and z

axes. Suppose an external pressure, $p_{\rm ext}$, is applied on both of the cube sides that are parallel to the x-y plane, which applies a force purely perpendicular to the surface. What is the resulting strain and shape the cube takes due to the applied pressure?

The external pressure applies an external force on the two sides to which it is applied that is directed towards the interior of the cube. Keeping in mind that our conventions choose $\mathbf{n}$ to be an outward pointing unit normal this means the stress at the surface of application is $\tau^{zz} = -p_{\rm ext}$. Because the resulting force is normal to the surface we also know $\tau^{zx} = \tau^{zy} = 0$. Since the internal stress is assumed to be constant these are also the expressions for τ^{zz}, τ^{zx} and τ^{zy} throughout the interior of the cube. An identical argument for the other sides (for which no force is applied) implies the other components of τ^{ij} also vanish.

The only nonzero component of stress therefore is $\tau^{zz} = -p_{\rm ext}$. The fractional change in volume of the cube is given by $\mathrm{tr}\,\sigma = \nabla \cdot \mathbf{u}$ and because of (9.2.25) this is given by

$$\nabla \cdot \mathbf{u} = \frac{\mathrm{tr}\,\tau}{3\kappa} = \frac{\tau^{zz}}{3\kappa} = -\frac{p_{\rm ext}}{3\kappa}\,. \tag{9.2.27}$$

This being negative shows that the cube's volume is reduced (as might have been expected).

The total stress is given by (9.2.26) and so is

$$\sigma_{ij} = 0 \quad \text{if } i \neq j\,, \quad \sigma_{xx} = \sigma_{yy} = p_{\rm ex}\left(\frac{1}{6\mu} - \frac{1}{9\kappa}\right)\,, \quad \text{and} \quad \sigma_{zz} = -p_{\rm ex}\left(\frac{1}{3\mu} + \frac{1}{9\kappa}\right)\,. \tag{9.2.28}$$

The vanishing of σ_{ij} for $i \neq j$ implies

$$\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} = 0\,, \tag{9.2.29}$$

while the diagonal elements of σ_{ij} imply

$$\frac{\partial u_y}{\partial y} = \frac{\partial u_x}{\partial x} = p_{\rm ex}\left(\frac{1}{6\mu} - \frac{1}{9\kappa}\right) \quad \text{and} \quad \frac{\partial u_z}{\partial z} = -p_{\rm ex}\left(\frac{1}{3\mu} + \frac{1}{9\kappa}\right)\,. \tag{9.2.30}$$

These last equations integrate to give

$$\begin{aligned} u_x &= x p_{\rm ex}\left(\frac{1}{6\mu} - \frac{1}{9\kappa}\right) + f_x(y,z)\,, \\ u_y &= y p_{\rm ex}\left(\frac{1}{6\mu} - \frac{1}{9\kappa}\right) + f_y(x,z) \\ \text{and} \quad u_z &= -z p_{\rm ex}\left(\frac{1}{3\mu} + \frac{1}{9\kappa}\right) + f_z(x,y)\,, \end{aligned} \tag{9.2.31}$$

for integration functions f_x, f_y and f_z, each depending on the two indicated arguments. These functions must satisfy (9.2.29), and are otherwise chosen to satisfy the boundary conditions. Setting the centre of the cube at the orgin ($x = y = z = 0$) and demanding $\mathbf{u} = 0$ there then fixes the integration constants to all vanish.

For a cube whose initial sides were of length L the new lengths in the x and y and z directions are

$$L_x = L_y = L\left[1 + p_{\rm ex}\left(\frac{1}{6\mu} - \frac{1}{9\kappa}\right)\right] \quad \text{and} \quad L_z = L\left[1 - p_{\rm ex}\left(\frac{1}{3\mu} + \frac{1}{9\kappa}\right)\right]\,. \tag{9.2.32}$$

Notice the new volume is

$$L_x L_y L_z = L^3 \left[1 + 2p_{\rm ex}\left(\frac{1}{6\mu} - \frac{1}{9\kappa}\right)\right]^2 \left[1 - p_{\rm ex}\left(\frac{1}{3\mu} + \frac{1}{9\kappa}\right)\right] \simeq L^3 \left(1 - \frac{p_{\rm ex}}{3\kappa}\right), \tag{9.2.33}$$

in agreement with (9.2.27).

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9.2.4 Elastic Waves

We are now in a position to write down what Newton's 2nd law for the elastic medium in the small-displacement Hooke's Law regime implies for the displacement field $\mathbf{u}(\mathbf{r}, t)$. To do so we insert (9.2.24) into (9.2.4), leading to

$$\partial_t(\rho\,\partial_t \mathbf{u}) = \mu \nabla^2 \mathbf{u} + (\mu + \lambda)\nabla(\nabla \cdot \mathbf{u}) \tag{9.2.34}$$

which uses

$$\partial^j \sigma_{ij} = \tfrac{1}{2}\Big(\partial_i \partial_j u^j + \nabla^2 u_i\Big) = \tfrac{1}{2}\Big[\nabla(\nabla\cdot\mathbf{u}) + \nabla^2 \mathbf{u}\Big]_i \tag{9.2.35}$$

and

$$\partial_i {\rm tr}\,\sigma = \partial_i \partial_j u^j = \Big[\nabla(\nabla\cdot\mathbf{u})\Big]_i. \tag{9.2.36}$$

For simplicity consider the case where ρ is constant. In that case eq. (9.2.34) describes waves moving through the medium, as can be seen by seeking solutions of the form

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{u}_0\, e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}}. \tag{9.2.37}$$

Substituting this into (9.2.34) implies

$$\Big(\omega^2 \rho - \mu \mathbf{k}^2\Big)\mathbf{u}_0 - (\mu + \lambda)\mathbf{k}(\mathbf{k}\cdot\mathbf{u}_0) = 0\,. \tag{9.2.38}$$

This has two types of solutions. There are two independent choices for $\mathbf{u}_0$ that are perpendicular to $\mathbf{k}$, and for these *transverse sound waves* (9.2.38) implies

$$\omega^2 = c_t^2 \mathbf{k}^2 \quad \text{with sound speed} \quad c_t = \sqrt{\frac{\mu}{\rho}}\,. \tag{9.2.39}$$

Alternatively we can choose $\mathbf{u}_0$ parallel to $\mathbf{k}$ in which case (9.2.38) implies the resulting *longitudinal sound wave* satisfies

$$\omega^2 = c_l^2 \mathbf{k}^2 \quad \text{with sound speed} \quad c_l = \sqrt{\frac{2\mu + \lambda}{\rho}}\,. \tag{9.2.40}$$

Three types of sound waves propagate through the elastic medium but the longitudinal and transverse waves move through it with different sound speeds. More general solutions are found by superposing these particular solutions.

9.3 Fluids

As a final application of continuum methods consider next fluids: media for which the constituent atoms do *not* have energetically preferred fixed positions relative to one another, even in equilibrium. Everyday liquids and gasses like air or water provide familiar examples of systems like this.

Although fluids could still be described using a displacement field $\mathbf{u}(\mathbf{r}, t)$ it is in practice not that useful to do so. One reason it is not useful is because the freedom for atoms to move and the absence of a preference for atoms to maintain an equilibrium relative separation means the displacement $\mathbf{u}(\mathbf{r}, t)$ need not remain small over time. A related reason displacement is less useful as a dynamical variable is because the internal energy is largely independent of the relative positions of the atoms in the medium.

For fluids the internal energy instead is dominated by the kinetic energy of the atomic motion as they move freely relative to the centre of mass. This makes the fluid velocity field $\mathbf{v}(\mathbf{r}, t)$ the more useful variable. In the continuum limit we continue to suppose our spatial resolution only allows us to measure small regions (volume elements) of volume $\mathrm{d}V$ whose linear size is much smaller than the scales of interest but also much larger than the spacings between the constituent atoms.

9.3.1 Conserved quantities

Since atoms within fluids can move over significant distances it is no longer true that a given volume element will contain a fixed number of atoms, so quantities like the number density $n(\mathbf{r}, t)$ of atoms or their mass density $\rho(\mathbf{r}, t)$ are no longer likely to remain approximately constant. But because the number of atoms is conserved the only way the number of atoms within a region can change is if atoms physically move into or out of it through its boundary.

To see what this means in a continuum language consider a fixed region $\mathcal{R}$ with boundary $\partial\mathcal{R}$ that is wholly immersed within the fluid. The total number of atoms within this region, $N(\mathcal{R})$, is given by integrating the number density $n(\mathbf{r}, t)$ over the region. So if the physical position and shape of $\mathcal{R}$ is not changing the rate of change of the number of atoms within $\mathcal{R}$ is given by

$$\partial_t N(\mathcal{R}) = \int_{\mathcal{R}} \partial_t n \, \mathrm{d}V \,. \tag{9.3.1}$$

But this must equal the rate with which fluid particles move through the boundary of $\mathcal{R}$. This is characterized by the local particle flux $\mathbf{j}(\mathbf{r}, t)$, defined so that the number of particles per unit time passing through a small element of surface area $\mathrm{d}S$ is $\mathbf{j} \cdot \mathbf{n} \, \mathrm{d}S$ where $\mathbf{n}$ is again the outward pointing unit normal to the surface element in question.

The total rate with which particles enter the region $\mathcal{R}$ therefore is given by an integral over its surface of the form

$$F = -\oint_{\partial\mathcal{R}} \mathbf{j} \cdot \mathbf{n} \, \mathrm{d}S \,. \tag{9.3.2}$$

The overall negative sign appears because $\mathbf{n}$ is outward pointing and so $\mathbf{j}\cdot\mathbf{n}$ is positive when the flux of particles is out of the region $\mathcal{R}$ rather than in. Since particles are neither created or destroyed the rate of increase of the number contained in (9.3.1) must equal the rate with which they enter through the boundaries, given in (9.3.2). That is,

$$\int_{\mathcal{R}} \partial_t n \,\mathrm{d}V + \oint_{\partial\mathcal{R}} \mathbf{j}\cdot\mathbf{n}\,\mathrm{d}S = 0\,. \tag{9.3.3}$$

This can be made into a local statement by converting the surface integral into a volume integral using Gauss' theorem, leading to the equivalent statement

$$\int_{\mathcal{R}} \Bigl(\partial_t n + \nabla\cdot\mathbf{j}\Bigr)\mathrm{d}V = 0\,. \tag{9.3.4}$$

Since this must be true for *any* region $\mathcal{R}$ within the fluid the integrand itself must vanish, leading to what is called the *continuity equation*:

$$\partial_t n + \nabla\cdot\mathbf{j} = 0\,, \tag{9.3.5}$$

as the local expression of conservation of particle number. (Compare (9.3.5) with (6.1.35) discussed earlier.)

More information is possible if we know why the particles move through the surface bounding $\mathcal{R}$ since this tells us how $\mathbf{j}$ depends on other variables. In the simplest case atoms move into and out of $\mathcal{R}$ because they are physically being carried along by the fluid flow. In this case the flux of particles past any particular surface element $\mathrm{d}S$ is given by the product of the fluid velocity and the number density of particles (per unit volume) of the fluid:

$$\mathbf{j} = n\,\mathbf{v}\,. \tag{9.3.6}$$

When this is so (9.3.5) becomes

$$\partial_t n + \nabla\cdot(n\,\mathbf{v}) = 0\,. \tag{9.3.7}$$

Another way to write (9.3.7) is

$$D_t n + n\,\nabla\cdot\mathbf{v} = 0\,, \tag{9.3.8}$$

where the *convective time derivative* is defined by

$$D_t = \partial_t + \mathbf{v}\cdot\nabla\,. \tag{9.3.9}$$

This derivative measures to total change with time seen by fluid elements that ride along with the flow. For any local function $f(\mathbf{r},t)$ the convective derivative $D_t f = \partial_t f + \mathbf{v}\cdot\nabla f$ is the sum of the explicit time variation of f and the change one sees due to motion with velocity $\mathbf{v}$ due to any spatial gradients ∇f.

With this interpretation for D_t expression (9.3.8) also has a simple interpretation. We saw in (9.2.11) that $\nabla\cdot\mathbf{u} = (\delta\,\mathrm{d}V)/\mathrm{d}V$ describes the fractional change in the volume of a local volume element as it is locally displaced through a distance $\mathbf{u}(\mathbf{r},t)$. So $\nabla\cdot\mathbf{v} = \partial_t[(\delta\,\mathrm{d}V)/\mathrm{d}V]$ is the time rate of change of the differential volume change of a collection of atoms displaced with a velocity $\mathbf{v} = \partial_t\mathbf{u}$. Writing (9.3.8) as

$$\frac{D_t n}{n} = -\nabla\cdot\mathbf{v} \tag{9.3.10}$$

then shows that the fractional rate of change of particle density along the flow is completely due to the change of volume the particles occupy. That is to say: the total number of particles does not change along the flow.

A similar story applies for the mass density, $\rho = m\,n$, of the fluid to the extent that the average mass of each atom does not change in time. Here m is the mass of each atom if they all have the same mass, or a suitable average of the masses if there is more than one type of atom present. In this case the flux of mass across the boundary of a region is given by $m\,n\,\mathbf{v} = \rho\,\mathbf{v}$ and so the local expression of conservation of mass becomes

$$\partial_t\rho + \nabla\cdot(\rho\,\mathbf{v}) = 0\,. \tag{9.3.11}$$

The quantity $\rho\,\mathbf{v} = m\,n\,\mathbf{v}$ plays another role as well, since it can also be interpreted as the local density of momentum (per unit volume). But momentum is another conserved thing (in the absence of external forces) and so the same argument as given above implies that the integral over $\mathcal{R}$ of its local rate of change, $\partial_t(\rho\mathbf{v})$, must equal the total flux of momentum that enters through the boundary $\partial\mathcal{R}$. This flux can be found by an identical argument as given above if applied separately for each component of momentum.

That is, the local density of the x component of momentum is $\rho\,v_x$ and so the local flux of this component of momentum is $\rho\,v_x\,\mathbf{v}$. A similar result is true for the y and z components of momentum and so flux appropriate to the momentum density $\rho\,v_i$ is given by the symmetric tensor with components $\rho\,v_i v_j$. The local expression of conservation of momentum – *i.e.* the analogue of (9.3.11) for momentum – is given by

$$\partial_t(\rho\,v_i) + \partial^j(\rho\,v_i v_j) = 0 \quad \text{(no applied forces)}\,. \tag{9.3.12}$$

Writing $\partial^j(\rho\,v_i v_j) = v_i\,\partial^j(\rho\,v_j) + \rho\,v_j\,\partial^j v_i$ and $\partial_t(\rho\,v_i) = \partial_t\rho\,v_i + \rho\,\partial_t v_i$ together with conservation of mass – *i.e.* eq. (9.3.11) – allows (9.3.12) to be written

$$\rho\Big[\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\Big] = \rho\,D_t\mathbf{v} = 0 \quad \text{(no applied forces)}\,. \tag{9.3.13}$$

This also has a simple physical interpretation since $\mathbf{v}$ is the density of momentum $\rho\mathbf{v}$ divided by the density of mass, and so is the local momentum per unit mass (as opposed to momentum per unit volume). Eq. (9.3.13) then says that momentum per unit mass (or momentum per atom) of a volume element is unchanged as one is carried along the flow.

But momentum is famously *not* conserved in the presence of forces and so the expression of Newton's second law for fluids uses the net force applied to a fluid element on the right-hand side of (9.3.13).

9.3.2 Navier Stokes Equations

To express Newton's law we therefore replace the right-hand side of (9.3.13) with the net force per unit volume applied to the fluid element. We consider two types of forces when doing so. The first of these consists of long-range body forces applied to the volume element. If the corresponding force per unit volume is denoted $\mathbf{F}_B$ (with the 'B' standing for 'body' force), then $\mathbf{F}_B$ should appear on the right-hand side of (9.3.13).

Concretely, for a gravitational field generated by a potential Φ the gravitational force per unit volume is $\mathbf{F}_g = -\rho\,\nabla\Phi$ (which reduces to $\mathbf{F}_g = \rho\,\mathbf{g}$ in the special case of a constant gravitational field). For an electrostatic force one would instead have $\mathbf{F}_E = \mathfrak{q}\,\mathbf{E}$ where $\mathfrak{q}(\mathbf{r},t)$ is the medium's net charge per unit volume and $\mathbf{E}(\mathbf{r},t)$ is the applied electric field.

The other type of force to be considered consists of the short-range contact forces that act between neighbouring fluid elements. The short range of these forces implies these can only act on the boundaries of the any particular region and as a result they can be written as a total derivative of the local stress tensor, with components $F_i = \partial^j \tau_{ij}$.

Adding these two types of applied forces on the right-hand side of (9.3.13) allows its components to be written

$$\rho\Big[\partial_t v_i + (\mathbf{v}\cdot\nabla)v_i\Big] = \rho\,D_t v_i = (F_B)_i + \partial^j \tau_{ij}\,. \tag{9.3.14}$$

For this to be useful we require an expression for the stress tensor τ^{ij} as a function of other properties of the fluid, like its pressure or the components, v_k, of its velocity. To this end it is useful to decompose τ_{ij} into a trace piece plus a traceless part

$$\tau_{ij} = -p\,\delta_{ij} + \Theta_{ij} \tag{9.3.15}$$

where $\mathrm{tr}\,\Theta = \delta^{ij}\Theta_{ij} = 0$. As we saw in the discussion around (9.1.13) the coefficient p hear can be interpreted as the fluid's pressure. The sign is chosen so that p is the externally applied pressure (and so produces a force into the medium).

Θ_{ij} contains the contribution of forces that arise because of the variation of the fluid motion in space, so we seek its explicit dependence on the components of $\mathbf{v}$. It is useful when doing so to recall that our focus is on macroscopic properties of the fluid that vary only over distances much longer than interatomic spacings, a limit for which spatial derivatives of $\mathbf{v}$ can be regarded as being very small. The leading contribution to τ_{ij} should involve the fewest possible derivatives of $\mathbf{v}$.

It is tempting to believe that this means the leading term involves no derivatives of $\mathbf{v}$ at all (such as if the traceless symmetric tensor Θ_{ij} were to be proportional to $v_i v_j - \frac{1}{3}\,v^2\,\delta_{ij}$. But any term like this would not be invariant under the replacement $v_i \to v_i + c_i$ for a constant vector $\mathbf{c}$, and so is not invariant under the Galilean transformations discussed in §1.6.

Invariance under Galilean transformations requires the components v_i to appear differentiated at least once. The most general form involving only a single derivative is

$$\Theta_{ij} = \eta \Big[\partial_i v_j + \partial_j v_i - \tfrac{2}{3}\, \delta_{ij}(\nabla\cdot\mathbf{v})\Big] + \zeta\,(\nabla\cdot\mathbf{v})\,\delta_{ij}\,, \tag{9.3.16}$$

where the coefficients η and ζ are regarded as being properties characteristic of the fluid. η is known as the coefficient of *shear viscosity* while ζ is the coefficient of *bulk viscosity*, and are characteristic properties of the fluid.

Combining everything, using (9.3.15) and (9.3.16) in the equation of motion (9.3.14) gives the expression of Newton's 2nd law for fluids

$$\rho\Big[\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\Big] = \mathbf{F}_B - \nabla p + \eta\nabla^2\mathbf{v} + \big(\zeta + \tfrac{1}{3}\eta\big)\,\nabla(\nabla\cdot\mathbf{v})\,, \tag{9.3.17}$$

which assumes η and ζ are spacetime independent (as is in practice often true to good approximation). Eqs. (9.3.17) are called the *Navier-Stokes* equations. The Navier-Stokes equations together with equation (9.3.11) expressing conservation of mass determine the evolution of ρ and $\mathbf{v}$ in time for viscous fluids, assuming the quantities η, ζ and p are all known as functions of ρ.

The required relations between η, ζ, p and ρ are normally dictated once a choice is made for a thermodynamic equation of state for the fluid of interest. The key assumption is that the fluid is locally in thermal equilibrium with a given thermodynamic free energy, whose differentiation provides a relation between the pressure p and mass density ρ at each position.

There are several important special cases of the Navier-Stokes equations that have broad applications.

- **Ideal fluid:** An ideal fluid is one for which the viscosity terms in (9.3.17) are negligible, in which case they reduce to what are called *Euler's equations*:

  $$\rho\Big[\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\Big] = \mathbf{F}_B - \nabla p \quad \text{(no viscosity)}\,. \tag{9.3.18}$$

  As we see below, viscosity introduces dissipation and friction into the fluid flow and so statements about energy conservation become much cleaner in the ideal-fluid limit. We return in §9.3.5 below to a more precise statement of when viscosities can be neglected.

- **Incompressible fluid:** An incompressible fluid is one for which ρ is effectively constant, as can sometimes be a reasonable approximation for liquids like water, for example. In this case the expression of mass conservation (9.3.11) simplifies to the statement

  $$\nabla\cdot\mathbf{v} = 0 \quad \text{(incompressible flow)}\,. \tag{9.3.19}$$

Using this in (9.3.17) then gives

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{\mathbf{F}_B}{\rho} - \nabla \left(\frac{p}{\rho}\right) + \nu \nabla^2 \mathbf{v} \quad \text{(incompressible flow)}\,, \tag{9.3.20}$$

where $\nu := \eta/\rho$ is called the *kinematic viscosity*.

When the flow is incompressible and the body forces satisfy $\mathbf{F}_B = -\rho \nabla \Phi$ then taking the curl of (9.3.20) and using the vector identity $\nabla \times \nabla = 0$ allows it to be rewritten to involve only $\mathbf{v}$:

$$\partial_t (\nabla \times \mathbf{v}) - \nabla \times [\mathbf{v} \times (\nabla \times \mathbf{v})] = \nu \nabla^2 (\nabla \times \mathbf{v}) \quad \text{(conservative incompressible flow)}\,, \tag{9.3.21}$$

where we use the vector identity

$$\nabla \left(\tfrac{1}{2}\, v^2\right) = \mathbf{v} \times (\nabla \times \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{v}\,. \tag{9.3.22}$$

- **Irrotational fluid:** An irrotational fluid is defined as one for which the *vorticity* vanishes: *i.e.* $\nabla \times \mathbf{v} = 0$. When this is true the velocity can locally be written as the gradient of a scalar, $\mathbf{v} = \nabla \psi$, for some *velocity potential* ψ. Whenever this is true it follows that the fluid circulation about any closed path C within the fluid vanishes:

$$\oint_C \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0 \quad \text{(irrotational)}\,, \tag{9.3.23}$$

  where $\mathrm{d}\mathbf{l} = \mathbf{t}\, \mathrm{d}s$ is proportional to the unit vector $\mathbf{t}$ tangent to the curve C, along which the differential arc length is $\mathrm{d}s$. If a fluid is both irrotational *and* incompressible then (9.3.19) implies the velocity potential ψ satisfies Laplace's equation

$$\nabla^2 \psi = 0 \quad \text{(irrotational and incompressible).} \tag{9.3.24}$$

The Navier-Stokes equations describe a rich variety of phenomena whose full exploration goes well beyond the scope of these notes. The remainder of this section restricts itself to exploring a few representative and simple examples.

9.3.3 Hydrostatics

The simplest situation is when the fluid is not moving so $\mathbf{v} = 0$ everywhere. In this case the continuity equation (9.3.11) and the Navier-Stokes equations reduce to

$$\partial_t \rho = 0 \quad \text{and} \quad \nabla p = \mathbf{F}_B\,. \tag{9.3.25}$$

The first of these just says that conservation of mass implies the amount of mass per unit volume cannot change if the fluid is not moving. The second of these says that the force that arises because of a gradient in pressure must balance any applied body forces if the fluid is not to move. In particular the pressure must be constant if no body forces are present.

Suppose next that a body force is present of the form $\mathbf{F}_B = -\rho\,\nabla\Phi$ for some potential Φ. This form assumes the force is conservative and that its strength is proportional to the local mass density. This is true in particular for gravitational forces and it is also true for fictitious centrifugal and coriolis forces in non-inertial frames (such as for frames that rotate with the rotation of the Earth). It is even true for electrostatic forces if the fluid's charge to mass ratio $\mathfrak{q}/\rho$ is independent of position (see *e.g.* eq. (9.1.4)). In any of these situations the force-balance equation of (9.3.25) then becomes

$$\frac{1}{\rho}\,\nabla p + \nabla\Phi = 0\,. \tag{9.3.26}$$

More precise statements require knowledge of how p is related to ρ. For incompressible fluids, whose density is a constant regardless of pressure, eq. (9.3.26) implies $\nabla(p + \rho\,\Phi) = 0$ and so

$$p + \rho\,\Phi = \text{constant}\,. \tag{9.3.27}$$

This states that all equipotential surfaces (along which Φ is constant) are also surfaces of constant pressure. In the special case of a constant gravitational field $\mathbf{F}_g = \rho\,\mathbf{g}$ with $\mathbf{g} = -g\,\mathbf{e}_z$ pointing in the z direction then $\Phi = \Phi_0 + g(z - z_0)$ and so

$$p(z) = p_0 - \rho\,g(z - z_0)\,. \tag{9.3.28}$$

Surfaces of constant pressure (and in particular any surface of interface with another fluid like air) must in this case be surfaces of constant z (that is to say: horizontal). And the pressure of the incompressible fluid increases in a calculable way as a function of depth.

If the fluid is not incompressible (such as is the case, say, for air) then progress can still be made if it is assumed to be in thermal equilibrium because this implies a relationship between ρ and p. For instance for an ideal gas the ideal gas law states $p = nT$, where n is the gas's number density (in units for which Boltzmann's constant is $k_B = 1$). In this case $p/\rho = T/m$ and we use $\rho = m\,n$ where m is the average atomic mass and so (9.3.26) implies

$$T(\mathbf{r}, t) + m\,\Phi(\mathbf{r}, t) = \text{constant} \quad \text{(ideal gas)}\,. \tag{9.3.29}$$

For a constant gravitational field $\Phi = \Phi_0 + g(z - z_0)$ this implies

$$T(z) = T_0 - mg(z - z_0)\,, \tag{9.3.30}$$

showing how temperature falls with height for an equilibrium fluid in a constant gravitational field.

If the fluid is instead held at constant temperature then it is useful to think in terms of the fluid's Gibbs free energy per unit mass, $\mathfrak{g}(T, p)$, whose natural variables are temperature T and pressure p, with standard thermodynamics arguments implying

$$d\mathfrak{g} = -\mathfrak{s}\, dT + \mathfrak{v}\, dp\,, \tag{9.3.31}$$

where $\mathfrak{s}(\mathbf{r}, t)$ here denotes the local entropy per unit mass, in terms of which the entropy per unit volume discussed in previous sections is $s = \rho\,\mathfrak{s}$. The variable $\mathfrak{v}(\mathbf{r}, t) = 1/\rho$ is the volume per unit mass of the fluid. For isothermal situations where $dT = 0$ this shows $d\mathfrak{g} = \mathfrak{v}\, dp = dp/\rho$ and so (9.3.26) implies

$$\nabla\Big(\mathfrak{g} + \Phi\Big) = 0 \quad \text{(isothermal)}\,. \tag{9.3.32}$$

Equipotential surfaces along which $\Phi(\mathbf{r})$ is constant also must be surfaces of constant $\mathfrak{g}(\mathbf{r})$.

Alternatively, if there is negligible heat exchange within the fluid then the fluid profile is *adiabatic* instead of isothermal. In this case its entropy per unit mass does not change and so its entropy per unit volume $s = \rho\,\mathfrak{s}$ is conserved (must satisfy a continuity equation):

$$\partial_t(\rho\,\mathfrak{s}) + \nabla\cdot(\rho\,\mathfrak{s}\,\mathbf{v}) = 0\,. \tag{9.3.33}$$

Together with the continuity equation for mass – eq. (9.3.11) – this implies

$$D_t\mathfrak{s} = \partial_t\mathfrak{s} + \mathbf{v}\cdot\nabla\mathfrak{s} = 0\,. \tag{9.3.34}$$

For static fluids this implies in particular $\partial_t\mathfrak{s} = 0$.

For the present purposes what is important is that being at constant entropy again allows the combination $\nabla p/\rho$ to be written as a gradient. This is because the fluid's specific enthalpy density, $\mathfrak{w}(\mathbf{r}, t) = \mathfrak{w}[\mathfrak{s}(\mathbf{r}, t), \mathfrak{v}(\mathbf{r}, t)]$, has as its natural variables the entropy per unit mass, $\mathfrak{s}$, and the pressure: $d\mathfrak{w} = T d\mathfrak{s} + \mathfrak{v}\, dp$ with the volume per unit mass given by $\mathfrak{v} = 1/\rho$. It follows that

$$(\nabla\mathfrak{w})_{\rm ad} = \frac{1}{\rho}\,\nabla p \tag{9.3.35}$$

and this allows (9.3.26) to be written

$$\nabla\Big(\mathfrak{w} + \Phi\Big) = 0 \quad \text{(adiabatic)}\,. \tag{9.3.36}$$

In all of these cases the conditions of hydrostatic equilibrium together with thermal equilibrium allow the response of fluid properties to applied body forces to be computed explicitly. In the case of a gravitational force different types of equilibrium give different types of density profiles for different fluids, but all three examples – *e.g.* eqs. (9.3.29), (9.3.32) and (9.3.36) – agree on at least one thing: all thermodynamic quantities are constant along equipotential surfaces. For a potential like $\Phi(z) = gz$ the pressure, temperature and density are all functions of z only. If one were to have a thermodynamic variable like temperature vary at fixed z in a gravitational field then the fluid configuration cannot be static. As applied to the atmosphere this is part of the reason for the existence of the phenomenon of wind.

Archimede's principle

The Navier-Stokes equations contain in particular many older observations about the properties of fluids. One such is Archimede's principle that provides a criterion for whether an object of mass M and volume V will float or sink when immersed within a fluid subject to a gravitational field. The principle states that the object floats (sinks) if its mass is less than (heavier than) the mass of the fluid that it displaces.

To see why this follows from (9.3.25) consider the balance of forces on the immersed object. The net force applied to the immersed object is the force of gravity and the sum of the fluid pressure applied to its surface:

$$\mathbf{F}_{\rm net} = M\,\mathbf{g} - \oint_{\partial\mathcal{R}} \mathbf{n}\,p\,{\rm d}S\,, \tag{9.3.37}$$

where $\mathbf{n}$ is (as usual) the unit outward-pointing normal vector and the integral is over $\partial\mathcal{R}$ (the boundary of the immersed object). The integral over the pressure is not zero because hydrostatic equilibrium implies the pressure of the fluid is higher at depth because its gradient must balance the force of gravity on the fluid.

Archimede's principle follows if one imagines removing the immersed object and replacing it with the fluid, in which case for constant gravitational field (9.3.25) implies

$$\nabla p = \mathbf{F}_{B} = \rho\,\mathbf{g}\,. \tag{9.3.38}$$

Integrating the left-hand side of this over the volume $\mathcal{R}$ the immersed object would have occupied then gives

$$\int_{\mathcal{R}} \nabla p\,{\rm d}V = \oint_{\partial\mathcal{R}} p\,\mathbf{n}\,{\rm d}S \tag{9.3.39}$$

on use of Gauss' theorem. Integrating the right-hand side of (9.3.38) instead gives $M_{\rm fl}(\mathcal{R})\,\mathbf{g}$ where $M_{\rm fl}(\mathcal{R})$ is the integral of the fluid density over the region $\mathcal{R}$ and so is the mass of fluid that the immersed body displaces.

So integrating (9.3.38) over $\mathcal{R}$ implies

$$\oint_{\partial\mathcal{R}} p\,\mathbf{n}\,{\rm d}S = M_{\rm fl}(\mathcal{R})\,\mathbf{g}\,. \tag{9.3.40}$$

Comparing this to (9.3.37) then shows that the net force on the immersed object can be written

$$\mathbf{F}_{\rm net} = \Big[M - M_{\rm fl}(\mathcal{R})\Big]\mathbf{g}\,, \tag{9.3.41}$$

and so the force is directed up if $M < M_{\rm fl}(\mathcal{R})$ and is directed down if $M > M_{\rm fl}(\mathcal{R})$, which is pricely what Archimede's principle would say. The object neither rises or sinks if $M = M_{\rm fl}(\mathcal{R})$.

Self-gravitating systems

A variant of the above discussion is the case where the gravitational field is sourced by the fluid itself (rather than being a given field that is applied from the outside). The gravitational potential is determined by solving Poisson's equation

$$\nabla^2 \Phi = 4\pi\rho \tag{9.3.42}$$

where ρ is the mass density responsible for the field. For self-gravitating fluids the gravitational field experienced is sourced by the fluid itself.

In such situations one solves for ρ, p and Φ together, using the equation of state $p(\rho)$, the fluid equation (9.3.26) and Poisson's equation (9.3.42). The gravitational field can be eliminated from these by taking the divergence of (9.3.26) and using (9.3.42) in the result, leading to the following condition for mechanical equilibirum:

$$\nabla \cdot \left(\frac{1}{\rho} \, \nabla p \right) = -\nabla^2 \Phi = -4\pi\rho \, . \tag{9.3.43}$$

For instance, for a spherically symmetric source it is useful to use spherical polar coordinates and choose $\rho = \rho(r)$ and $p = p(r)$. In this case (9.3.43) becomes

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left[\frac{r^2}{\rho} \left(\frac{\mathrm{d}p}{\mathrm{d}r} \right) \right] = -4\pi\rho \, . \tag{9.3.44}$$

The above derivation shows that this equation follows purely by requiring mechanical equilibrium and so does not depend at all on whether or not the fluid is in thermal or chemical equilibrium.

Equilibrium can enter by providing an equation of state $p(\rho)$, after which (9.3.44) becomes an ordinary differential equation that can be solved for $\rho(r)$. Once this is done both $p(r)$ and $\Phi(r)$ can then be determined using the equation of state and (9.3.42).

For spherically symmetric solutions the first derivative of ρ (or any other scalar) must vanish at the origin: $\rho'(0) = 0$. This means the solution found by integrating (9.3.44) is completely determined by the value ρ_0 chosen for $\rho(0)$. For the equations of state of physical interest $\rho(r)$ found by integrating (9.3.44) decreases monotonically with r, until eventually $\rho \to 0$ as $r \to R$. This is interpreted as the exterior of the self-gravitating object. The mass of the object is then found by integrating the solution $M = 4\pi \int_0^R \mathrm{d}r \, r^2 \rho(r)$.

This procedure predicts that for a given equation of state both M and R are determined by the single parameter ρ_0. This implies there must be a relationship $M = M(R)$ relating mass and radius for self-gravitating objects built from materials sharing the same equation of state. This is indeed borne out by observations of stars, for which *e.g.* Hydrogen burning 'main-sequence' stars fall along a line when plotted in the M-R plane, while *e.g.* 'red-giant' stars (that burn different nuclear fuels and so do not share the same equation of state) fall along a different line.

9.3.4 Steady Flow

Consider next situations where $\mathbf{v}$ is nonzero but time-independent. In this case the Navier-Stokes equation (9.3.17) becomes

$$\rho\,(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{F}_B - \nabla p + \eta\nabla^2\mathbf{v} + \left(\zeta + \tfrac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})\,, \tag{9.3.45}$$

For static flows the velocity field $\mathbf{v}(\mathbf{r})$ is tangent to the trajectories $\mathbf{r}(t)$ of fluid particles,

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \propto \mathbf{v} \quad \text{or, equivalently} \quad \frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y} = \frac{\mathrm{d}z}{v_z}\,. \tag{9.3.46}$$

Solutions $\mathbf{r}(t)$ to (9.3.46) for given $\mathbf{v}(\mathbf{r},t)$ are called *streamlines* of the fluid flow. For time-dependent $\mathbf{v}(\mathbf{r},t)$ the streamlines to which the condition (9.3.46) leads need not be the same as the trajectories of individual fluid particles. From this point of view streamlines provide a running snapshot of all particle trajectories at a given time, but any given particle need not remain on its initial streamline if $\mathbf{v}(\mathbf{r},t)$ depends on t.

Bernoulli's equation

It turns out that (9.3.45) in some circumstances implies a simple statement for how velocities vary along streamlines. To find out what this is we take its dot product with the streamline direction (the unit vector $\hat{\mathbf{v}}$ in the direction of $\mathbf{v}$). When doing so it is useful to specialize to body forces of the form $\mathbf{F}_B = -\rho\,\nabla\Phi$ as well as use the general vector identity (9.3.22). Using these in (9.3.45) leads to

$$\tfrac{1}{2}\nabla(v^2) - \mathbf{v}\times(\nabla\times\mathbf{v}) = -\frac{1}{\rho}\,\nabla p - \nabla\Phi + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \left(\frac{\zeta}{\rho} + \frac{\eta}{3\rho}\right)\nabla(\nabla\cdot\mathbf{v})\,. \tag{9.3.47}$$

Next take the dot product of $\hat{\mathbf{v}}$ with (9.3.47), keeping in mind that $\mathbf{v}\times(\nabla\times\mathbf{v})$ is perpendicular to $\hat{\mathbf{v}}$. One then finds

$$\hat{\mathbf{v}}\cdot\nabla\Big(\tfrac{1}{2}v^2 + \Phi\Big) + \frac{1}{\rho}\hat{\mathbf{v}}\cdot\nabla p = \hat{\mathbf{v}}\cdot\left[\frac{\eta}{\rho}\nabla^2\mathbf{v} + \left(\frac{\zeta}{\rho} + \frac{\eta}{3\rho}\right)\nabla(\nabla\cdot\mathbf{v})\right]. \tag{9.3.48}$$

This is an equation with which one could get emotionally involved if the left-hand side could be written as the gradient of something, ∇W. In this case $\hat{\mathbf{v}}\cdot\nabla W$ is just the rate of change of W along a streamline. W would in particular be a constant along the streamline if the viscosities on the right-hand side of (9.3.48) could be neglected. As mentioned earlier, neglect of the viscosities in the Navier-Stokes equation is called the ideal fluid limit – *c.f.* the discussion around eq. (9.3.18) – and it can be a good approximation in many real systems.

A simple case where the left-hand side of (9.3.48) is a gradient is when the flow is incompressible (such as is water in many applications). In this case ρ is constant and the left-hand side is

$$\hat{\mathbf{v}}\cdot\nabla\left(\tfrac{1}{2}v^2 + \frac{p}{\rho} + \Phi\right) \tag{9.3.49}$$

and so $\frac{1}{2}v^2+(p/\rho)+\Phi$ is conserved along a streamline if the fluid also has negligible viscosities.

Another case where the left-hand side of (9.3.48) is a gradient is when the flow is adiabatic so $\nabla p/\rho = \nabla w$ where w is the Gibbs energy per unit mass – *c.f.* eq. (9.3.35). In this case the left-hand side of (9.3.48) becomes

$$\hat{\mathbf{v}} \cdot \nabla\Big(\tfrac{1}{2}v^2 + w + \Phi\Big)\,, \tag{9.3.50}$$

and so $\frac{1}{2}v^2 + w + \Phi$ is conserved along streamlines if the viscosities can be neglected.

Conservation of the quantities appearing in (9.3.48) or (9.3.50) along a streamline is called *Bernoulli's equation*, and they each have the spirit of an energy conservation condition along a streamline. From that perspective the fact that viscosity obstructs this kind of conservation speaks to its interpretation as source of energy dissipation within the fluid.

Cyclones

As a variation on the current theme of steady flows we next consider the implications of the Navier-Stokes equations for long-range flows within the atmosphere. There are several peculiar things about the atmosphere when it is regarded as a fluid. First, the atmosphere is not that thick compared to its physical extent along the Earth's suface. This means that short-scale phenomena of size less than a few km are described by fluid that is free to move (under the influence of gravity) in all three spatial dimensions. The same is not true for phenomena on much larger scales because for these the motion is effectively only in the two dimensions parallel to the Earth.

It is also true that the surface of the Earth is not an inertial frame and so besides gravity the quantity Φ governing the properties of the body forces also includes the effects of the fictional centrifugal and coriolis forces. We explore one of the consequence of these fictitious forces in this section: their role in the formation of *cyclones* in the atmosphere.

The starting point is eq. (3.3.4) that expresses the acceleration experienced by a fluid particle as seen by a reference frame at the Earth's centre that rotates with the Earth. The upshot is to replace the local acceleration due to gravity, $\mathbf{g}$, with

$$\mathbf{g} - \boldsymbol{\Omega} \times \Big(\boldsymbol{\Omega} \times \mathbf{r}\Big) - 2\,\boldsymbol{\Omega} \times \frac{\partial \mathbf{r}}{\partial t}\,. \tag{9.3.51}$$

For applications to the Earth's atmosphere we can take the magnitude $|\mathbf{r}|$ of all fluid positions to be approximately the Earth's radius $R_\oplus$, while the angular coordinates keep track of the latitude and longitude of the fluid elements we consider. The velocity appearing in (9.3.51) can similarly be replaced by the fluid velocity $\mathbf{v}$ in the rotating frame.

From this point of view the effect of the Earth's rotation on fluid motion is to replace the gravitational body force $\mathbf{F}_B = \rho\,\mathbf{g}$ appearing in (9.3.17) with

$$\mathbf{F}_B = \rho\Big(\mathbf{g}_{\rm eff} - 2\,\boldsymbol{\Omega} \times \mathbf{v}\Big)\,, \tag{9.3.52}$$

where

$$\mathbf{g}_{\rm eff}(\theta,\phi) = \mathbf{g} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) = (-g + \Omega^2 R_\oplus \sin^2\theta)\,\mathbf{e}_r + \Omega^2 R_\oplus \sin\theta\cos\theta\,\mathbf{e}_\theta\,, \tag{9.3.53}$$

and $\boldsymbol{\Omega}(\theta,\phi) = \Omega\,\mathbf{e}_z$, where $\Omega = 7.292\times 10^{-5}$/s. (See (3.3.5) for the various sizes of the different terms.)

The equation governing steady atmospheric flow of an ideal fluid (no viscosity) in a reference frame rotating with the Earth is found by using (9.3.52) in (9.3.45) to get

$$\rho\,(\mathbf{v}\cdot\nabla)\mathbf{v} = \rho\Big(\mathbf{g}_{\rm eff} - 2\,\boldsymbol{\Omega}\times\mathbf{v}\Big) - \nabla p\,, \tag{9.3.54}$$

We seek a steady flow solution of these equations that describes a *cyclone*: a circular pattern of flow around a region of low or high pressure whose radius is much smaller than the Earth's radius but possibly much wider than the height of the atmosphere.

To this end suppose we denote by x and y the east-west and north-south directions near a particular latitude and longitude (θ,ϕ) on the Earth's surface, with $x = y = 0$ being a local maximum or minimum of pressure. Define polar coordinates on the surface of the Earth around this point, with $x = \varrho\cos\vartheta$ and $y = \varrho\sin\vartheta$, and define the unit vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ pointing along these two directions. Because these are both parallel to the Earth's surface they can be expressed in terms of the spherical polar coordinates, (r,θ,ϕ), used in §3.3, whose origin is at the Earth's centre and whose z axis points towards the North Pole. In particular $\mathbf{e}_x = \mathbf{e}_\phi$ points in the direction of increasing longitude while $\mathbf{e}_y = -\mathbf{e}_\theta$ points in the direction of increasing (decreasing) latitude in the northern (southern) hemisphere.

In these coordinates we imagine there to be an atmospheric pressure gradient, $p = p(\varrho)$, that is just a function of the radial coordinate ϱ. The derivative $p' = {\rm d}p/{\rm d}\rho$ is positive for a low-pressure region and p' is negative for a high-pressure region.

To find a solution with the fluid velocity moving in a circle around the centre of high or low pressure we take

$$\mathbf{v} = v(\varrho)\,\mathbf{e}_\vartheta = -v(\varrho)\,\sin\vartheta\,\mathbf{e}_x + v(\varrho)\,\cos\vartheta\,\mathbf{e}_y\,. \tag{9.3.55}$$

This and the choice $p = p(\varrho)$ implies in particular that $\nabla p \propto \mathbf{e}_\varrho$ is perpendicular to $\mathbf{v}\propto\mathbf{e}_\vartheta$ and so the winds move along lines of constant pressure. This is by contrast with the situation without the coriolis force, for which forces generated by the pressure gradient produce motion in the direction of $-\nabla p$, and so is perpendicular to the lines of constant pressure. Wind initially rushing radially in to a low-pressure region gets deflected to the side by the coriolis force, with the cyclone configuration (9.3.55) corresponding to a steady flow where the pressure gradient and coriolis forces are parallel, but acting in the opposite directions.

The choice (9.3.55) implies $v > 0$ gives flow towards increasing ϑ and so represents a counter-clockwise cyclone (seen from above) while $v < 0$ similarly implies clockwise flow (seen from above). Using the expression for the gradient in polar coordinates $\nabla = \mathbf{e}_\varrho\,\partial_\varrho + (\mathbf{e}_\vartheta/\varrho)\,\partial_\vartheta$ together with (9.3.55) implies

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{v}{\varrho}\left(\frac{\partial \mathbf{v}}{\partial\vartheta}\right) = -\frac{v^2}{\varrho}\Big(\cos\vartheta\,\mathbf{e}_x + \sin\vartheta\,\mathbf{e}_y\Big) = -\frac{v^2}{\varrho}\,\mathbf{e}_\varrho\,. \tag{9.3.56}$$

Unsurprisingly, with the choices made above the acceleration $D_t\mathbf{v} = \partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}$ gives the centripetal acceleration required for circular motion.

We may neglect the component of $\mathbf{g}_{\rm eff}$ that is tangent to the Earth's surface[31] and seek the components of the pressure and coriolis forces tangent to the Earth's surface,

$$\nabla p = \frac{\mathrm{d}p}{\mathrm{d}\varrho}\,\mathbf{e}_\varrho = p'(\varrho)\,\mathbf{e}_\varrho\,, \tag{9.3.57}$$

and the horizontal component of the coriolis force becomes

$$\begin{aligned} -2(\boldsymbol{\Omega}\times\mathbf{v})_{\rm hor} &= -2v\Omega(\mathbf{e}_z\times\mathbf{e}_\vartheta)_{\rm hor} = 2v\Omega\Big[(\mathbf{e}_z\times\mathbf{e}_\phi)_{\rm hor}\sin\vartheta + (\mathbf{e}_z\times\mathbf{e}_\theta)_{\rm hor}\,\cos\vartheta\Big] \\ &= 2v\Omega\cos\theta_0(\mathbf{e}_x\,\cos\vartheta + \mathbf{e}_y\,\sin\vartheta) = 2v\Omega\cos\theta\,\mathbf{e}_\varrho\,, \end{aligned} \tag{9.3.58}$$

where the cross products in these expressions are evaluated using (3.3.15) together with the identifications $\mathbf{e}_x = \mathbf{e}_\phi$ and $\mathbf{e}_y = -\mathbf{e}_\theta$.

Combining everything, each term in eq. (9.3.54) points in the $\mathbf{e}_\varrho$ direction, leading to the single equation relating v to p'/ρ

$$\frac{v^2}{\varrho} = \frac{p'}{\rho} - 2v\Omega\cos\theta_0\,, \tag{9.3.59}$$

This expression must hold for all $\varrho \geq 0$ and this is what determines the radial wind profile within the cyclone, given a known pressure profile. At the centre the pressure distribution can only remain smooth if $p' \to 0$ as $\varrho \to 0$, and so (9.3.59) implies that v also vanishes at the centre. This is true no matter how large v is for large ϱ, and is the origin of the 'eye' of a hurricane.

Far from the centre of the cyclone we can take ϱ large and so drop the left-hand side of (9.3.59), implying

$$\frac{p'}{\rho} \simeq 2v\Omega\cos\theta_0 \quad \text{(far from cyclone's centre)}. \tag{9.3.60}$$

It is clear that for this to work the sign of v must be correlated with the sign of p', with the correlation differing in the northern and southern hemispheres (since $\cos\theta_0 > 0$ in the north and $\cos\theta_0 < 0$ in the south). In the northern hemisphere counter-clockwise motion ($v > 0$) goes with a low-pressure region ($p' > 0$) and clockwise motion ($v < 0$) similarly goes with a high-pressure region ($p' < 0$). The correlation is opposite in the southern hemisphere.

[31] The discussion in §3.3.1 shows that tangential component of the centrifugal force points south in the northern hemisphere and north in the southern hemisphere and vanishes at the equator and at the north and south pole, so its neglect is a good approximation at the near-equatorial latitudes relevant for most cyclones.

For general ϱ solving (9.3.59) for v gives

$$v = -\Omega \cos\theta_0 \, \varrho \pm \sqrt{\Omega^2 \cos^2\theta_0 \varrho^2 + \frac{\varrho\, p'}{\rho}} \tag{9.3.61}$$

where the sign of the root is to be chosen to ensure $v \to 0$ when $p' \to 0$ (and so depends on the sign of $\Omega \cos\theta_0$). For low-pressure regions the square root is always real because $p' > 0$ and so in principle the larger the pressure gradient the larger the wind speed. This is why hurricanes form around low-pressure regions. For high-pressure regions by contrast we have $p' < 0$ and so increasing $|p'|$ eventually leads to the square root becoming imaginary, in which case no real solution exists for v. This is why hurricanes do not form around high-pressure regions.

High and low pressure zones differ because circular flow requires the centripetal acceleration to be directed inward and the size of the required acceleration grows like the square of v. For low-pressure regions the pressure gradient points inward while the coriolis force points outward, so higher pressure gradients can drive the required centripetal acceleration. For high-pressure regions the pressure gradient points outward and the coriolis force points inward, but the coriolis force only grows linearly with v and so cannot provide sufficient force when the wind speed is too large.

9.3.5 Boundary conditions and viscosity

The Navier-Stokes equations (supplemented by the continuity equation and any thermodynamic relations) play the role of Newton's 2nd law and so provide the evolution equations required to evolve any particular fluid configuration forward in time. But a well posed prediction also requires boundary conditions to be specified, such as on surfaces when a fluid comes into contact with other objects (like the edge of a container or the interface between water and the atmosphere).

The boundary conditions we use turn out to depend in an important way on whether or not the viscosity terms can be neglected. To see why, recall that viscosity introduces dissipation and friction into the fluid's motion, in the sense that viscous terms obstruct Bernoulli's equation expressing energy conservation along a fluid streamline. In this sense ideal fluids can be thought of as being 'friction free', and the boundary condition usually imposed at a solid surface when using ideal fluids asks that the fluid velocity be tangent to the surface: *i.e.*

$$v_n = \mathbf{v} \cdot \mathbf{n} = v_n^{\text{container}} \quad \text{on boundary (ideal fluid)}\,, \tag{9.3.62}$$

where $\mathbf{n}$ is a unit vector normal to the surface. This boundary condition forbids the fluid from penetrating into the container that surrounds it, while slipping along it without friction.

For viscous fluids a different boundary condition is used: the fluid velocity is asked to equal the velocity of the surrounding container, so that the *relative* velocity of fluid and container vanishes:

$$v_n = \mathbf{v} \cdot \mathbf{n} = v_n^{\rm container} \quad \hbox{and} \quad v_t = v_t^{\rm container} \quad \hbox{on boundary (viscous fluid)}\,, \tag{9.3.63}$$

and so in particular *both* components should vanish if the container is not moving. Physically this kind of boundary condition arises because of the attractive short-ranged inter-atomic forces at play between the fluid molecules and those within the container. In particular this constrains both the normal *and* tangential components of $\mathbf{v}$ at the container interface.

It might seem odd to have a different number of boundary conditions in these two cases, but it nonetheless makes sense mathematically because $\mathbf{v}$ only appears with single spatial derivatives in the ideal case, but is second-order in spatial derivatives in the viscous case (and so needs more boundary conditions to fix all of the integration constants).

Historically there was considerable debate about which of these boundary conditions should hold for real fluids, with the debate partly driven by the fact that measurements showed that for some real fluids flow velocities actually did seem to satisfy (9.3.63), such as at the walls for the flow of a fluid through a pipe. Yet sometimes the very same fluids seemed to be well-described as ideal fluids that slide along their container walls, subject to (9.3.62). What could be going on?

The answer turned out to be the existence of *boundary layers*. In real fluids (9.3.63) applies but this can sometimes be masked because the fluid transitions from full flow to zero flow only through a comparatively narrow boundary layer, with the fluid outside the boundary layer behaving much like an ideal fluid. In such fluids viscosity only plays an important role near the boundary. This section sketches how one tells when fluid viscosity can be neglected and when it cannot.

We start with the Navier-Stokes equations for an incompressible flow, including viscosity, since very viscous fluids are in practice often also incompressible. For these the mass density is constant and so the continuity equation (9.3.11) becomes $\nabla \cdot \mathbf{v} = 0$. The Navier-Stokes equations themselves become (in the absence of body forces) (9.3.20), repeated here for convenience of reference:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla \left(\frac{p}{\rho} \right) + \nu \nabla^2 \mathbf{v}\,, \tag{9.3.64}$$

where $\nu := \eta/\rho$ is the kinematic viscosity.

The key observation is that the $(\mathbf{v} \cdot \nabla)\mathbf{v}$ term scales differently than does the $\nabla^2 \mathbf{v}$ term, since the former is quadratic in $\mathbf{v}$ and linear in derivatives while the latter is linear in $\mathbf{v}$ and quadratic in derivatives. These two terms involve only a single parameter ν, which has dimensions of $(\hbox{length})^2/(\hbox{time})$.

In any particular application the solution $\mathbf{v}(\mathbf{r}, t)$ acquires a dependence on the characteristic scales of the problem through the boundary conditions. (Perhaps one computes the flow pattern around an object of linear size L moving at a speed u relative to the fluid.) Keeping in mind that p/ρ has dimensions of velocity squared, every term in (9.3.68) has dimension velocity/time, or equivalently (velocity)^2/(length).

If one scales out the characteristic length scale L and speed u by defining dimensionless variables $\hat{\mathbf{v}}$, $\hat{\mathbf{r}}$ and $\hat{t}$ through the definitions

$$\mathbf{v} = u\,\hat{\mathbf{v}}\,, \quad \mathbf{r} = L\hat{\mathbf{r}} \quad \text{and} \quad t = L\hat{t}/u\,, \tag{9.3.65}$$

then (9.3.68) becomes

$$\frac{u^2}{L}\left[\hat{\partial}_t\hat{\mathbf{v}} + (\hat{\mathbf{v}}\cdot\hat{\nabla})\hat{\mathbf{v}} + \hat{\nabla}\left(\frac{\hat{p}}{\hat{\rho}}\right) - \frac{1}{R}\hat{\nabla}^2\hat{\mathbf{v}}\right] = 0\,, \tag{9.3.66}$$

and so any solution $\hat{\mathbf{v}}(\hat{\mathbf{r}},\hat{t})$ depends on ν only through the dimensionless combination called the *Reynold's number*,

$$R := \frac{uL}{\nu}\,. \tag{9.3.67}$$

The importance of the viscosity term depends on the size of R for any particular application. In particular, the viscosity term can be neglected relative to the convective $(\mathbf{v}\cdot\nabla)\mathbf{v}$ term in the limit of large Reynold's number $R \gg 1$, leading to the perfect fluid limit:

$$\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} \simeq -\nabla\left(\frac{p}{\rho}\right)\,. \tag{9.3.68}$$

In the opposite case of low Reynold's number, $R \ll 1$, it is the convective term that can be dropped, leading to the approximate linear evolution equation

$$\partial_t\mathbf{v} \simeq -\nabla\left(\frac{p}{\rho}\right) + \nu\nabla^2\mathbf{v}\,, \tag{9.3.69}$$

The linearity of this system allows (9.3.69) to be explicitly solved in many interesting situations, despite viscosity playing an important role.

Inspection of (9.3.67) shows that small R occurs when speeds are small and distances are short. But the boundary condition (9.3.63) makes this always true near a stationary boundary since the boundary conditions set $\mathbf{v} \to 0$ there. This is why viscosity can dominate the behaviour near boundaries even while being negligible further into the fluid's bulk.

9.3.6 Time-dependent flows

This section closes with a discussion of time-dependent phenomena, most notably wave propagation within fluids. Besides providing a simple illustration of time-dependent methods, a discussion of waves helps underline the ubiquity of wave phenomena in continua and also to highlight how transverse waves are harder to to support in fluids due to the absence of shear stresses.

Small-amplitude Waves

To study waves in a fluid consider an adiabatic ideal fluid, for which the equations of motion are (9.3.11):

$$\partial_t \rho + \nabla \cdot (\rho \cdot \mathbf{v}) = 0 \tag{9.3.70}$$

and the Navier-Stokes equation in its form (9.3.18) in the absence of body forces:

$$\rho \Big[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} \Big] = -\nabla p \,. \tag{9.3.71}$$

We seek solutions to these equations by perturbing about a static solution. That is, we write

$$\rho = \rho_0 + \hat{\rho} \,, \quad p = p_0 + \hat{p} \tag{9.3.72}$$

where ρ_0 and p_0 satisfy (9.3.70) and (9.3.71) with $\mathbf{v} = 0$, which implies ρ_0 and p_0 are both constants, related to one another by the fluid's equation of state $p = p(\rho)$. We wish to solve for the quantities $\hat{\rho}(\mathbf{r}, t)$ and $\hat{p}(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$ under the assumption that these are all very small, so that (9.3.70) and (9.3.71) can be expanded in these variables, keeping only the linear terms to first approximation.

In this case the linearized equations become

$$\partial_t \hat{\rho} + \rho_0 \nabla \cdot \mathbf{v} \simeq 0 \tag{9.3.73}$$

and

$$\rho_0 \, \partial_t \mathbf{v} \simeq -\nabla \hat{p} = -\left(\frac{\mathrm{d}p}{\mathrm{d}\rho} \right)_{\mathrm{ad}} \nabla \hat{\rho} = -c_s^2 \, \nabla \hat{\rho} \,, \tag{9.3.74}$$

where the last equality defines $c_s^2 = (\mathrm{d}p/\mathrm{d}\rho)_{\mathrm{ad}}$ and the subscript 'ad' indicates that the derivative $\mathrm{d}p/\mathrm{d}\rho$ is performed with the entropy density fixed. The velocity $\mathbf{v}$ can be eliminated between these two equations by taking the time derivative of (9.3.73) and subtracting from it the divergence of (9.3.74), leading to

$$\partial_t^2 \hat{\rho} = -\rho_0 \nabla \cdot \partial_t \mathbf{v} = c_s^2 \nabla^2 \hat{\rho} \,. \tag{9.3.75}$$

This shows that the pressure fluctuation $\hat{\rho}$ satisfies the wave equation $-\partial_t^2 \hat{\rho} + c_s^2 \nabla^2 \hat{\rho} = 0$ with wave propagation speed c_s (compare with (5.3.16)). Eq. (9.3.73) then shows that the compression mode $\nabla \cdot \mathbf{v}$ of the velocity field also satisfies a similar equation with the same propagation speed. Because $\nabla \cdot \mathbf{v}$ describes a local change in the volume of the fluid – *c.f.* the discussion around (9.3.10) – the joint oscillations of $\hat{\rho}$ and $\nabla \cdot \mathbf{v}$ describe a compression wave that passes through the fluid.

What about the transverse components of the fluid? The velocity field has three components and $\nabla \cdot \mathbf{v}$ only tracks one combination of these. The other two can be chosen to satisfy $\nabla \cdot \mathbf{v} = 0$ and are called the transverse part of the fluid because if one seeks a wave solution of the form $\mathbf{v} = \mathbf{v}_0 \, e^{-i\omega t + i\mathbf{k} \cdot \mathbf{r}}$ then $\nabla \cdot \mathbf{v} = 0$ implies $\mathbf{v}_0 \cdot \mathbf{k} = 0$. This part of the velocity field contributes to its curl, $\nabla \times \mathbf{v}$ rather than its divergence, so its evolution can be found by taking the curl of (9.3.74). This leads to

$$\partial_t(\nabla \times \mathbf{v}) = 0\,, \tag{9.3.76}$$

showing that the transverse components of the fluid velocity do not oscillate. They do not do so because the absence of shear stress in a fluid means there is no restoring force for these modes that can drive an oscillation.

10 Classical Fields

This section explores the situation where the dynamical variables of interest are fields $\psi^a(\mathbf{r}, t)$ defined throughout space and time rather than a discrete collection of variables $q^A(t)$.

The most basic comparison between fields and dynamical variables in classical mechanics suggests that for fields the role of the index 'A' is partially played by spatial position $\mathbf{r}$, in addition to the discrete index a that distinguishes multiple fields from one another. This means that any instances where A is summed over must be replaced by a sum over a *and* an integral over position.

10.1 Scalar fields

To see how this works in detail, consider a single scalar field $\psi(\mathbf{r}, t)$. The assumption that $\psi(\mathbf{r}, t)$ is a scalar (as opposed to a vector or tensor) field just means that it returns a single number for every position in space and time (much like a varying temperature field $T(\mathbf{r}, t)$ might do). This is a simplifying assumption that is dropped once we look at other fields, like electromagnetism, in later sections.

10.1.1 Lagrangian formulation

The Lagrangian $L[\psi(\mathbf{r}, t), \partial_t\psi(\mathbf{r}, t)]$, for this field is defined so that its variation reproduces the field equation satisfied by $\psi(\mathbf{r}, t)$. In practical examples this field equation is local in the sense that it involves the field and its derivatives all evaluated at the same point, such as

$$\partial_t^2\psi(\mathbf{r}, t) = c_s^2\, \nabla^2\psi(\mathbf{r}, t) - V(\psi)\,, \tag{10.1.1}$$

for example, where c_s is a constant. The point is that $\partial_t^2\psi$ and $\nabla^2\psi$ are both evaluated at the same positions and times and $V(\psi)$ is an ordinary function of $\psi(\mathbf{r}, t)$ – such as $V(\psi) = \frac{1}{2}\, m^2\psi^2$ or something more complicated – with its argument also evaluated at the same position and time.

This kind of field equation arises if the Lagrangian comes as a local integral over space of a function of the field and its derivatives all evaluated at the same spacetime point:

$$L[\psi, \dot{\psi}] := \int \mathrm{d}^3x\; \mathcal{L}(\psi, \partial_t\psi, \nabla\psi, \cdots)\,, \tag{10.1.2}$$

where d^3x is just a shorthand for the spatial volume measure $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Requiring L to be a 'sum' over position in this way is similar to the way that the Lagrangian for independent physical systems $\{q^a\}$ and $\{Q^\alpha\}$ is obtained by adding the Lagrangian for the subsystems:[32] $L(q, Q, \dot{q}, \dot{Q}) = L(q, \dot{q}) + L(Q, \dot{Q})$.

The integrand $\mathcal{L}$ is called the system's *Lagrangian density*. For slowly varying fields it suffices to restrict to functions $\mathcal{L}$ that depend only on undifferentiated and singly differentiated fields while neglecting any dependence on multiple derivatives like $\partial_t^2\psi$ or $\nabla^4\psi$.

The action obtained from such a Lagrangian is then given by the spacetime integral of the Lagrangian density

$$S[\psi] = \int \mathrm{d}t\, L = \int \mathrm{d}^4x\; \mathcal{L}(\psi, \partial_t\psi, \nabla\psi)\,. \tag{10.1.3}$$

This way of writing the action is particularly convenient in relativistic theories because in this case the measure $\mathrm{d}^4x = \mathrm{d}t\,\mathrm{d}^3x = \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is Lorentz invariant and so the same must also be true for $\mathcal{L}$ (unlike, say, L or H). This makes its form easier to guess than for the Lagrangian or the Hamiltonian directly.

Given a Lagrangian density $\mathcal{L}$ we solve the variational problem as before: take the difference, $\delta S[\psi] = S[\psi + \delta\psi] - S[\psi]$ between two neighbouring configurations ψ and $\psi + \delta\psi$ and demand that its linear term in $\delta\psi$ vanish for arbitrary $\delta\psi$. For instance, varying (10.1.3) gives

$$\begin{aligned} \delta S &= \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial\psi}\,\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}\,\partial_t\delta\psi + \frac{\partial\mathcal{L}}{\partial\nabla\psi}\cdot\nabla\delta\psi\right] \\ &= \delta S_{\text{s.t.}} + \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial\psi} - \partial_t\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}\right) - \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial\nabla\psi}\right)\right]\delta\psi \end{aligned} \tag{10.1.4}$$

where $\delta S_{\text{s.t.}}$ denotes the 'surface terms' obtained by integrating by parts in space and time:

$$\begin{aligned} \delta S_{\text{s.t.}} &= \int \mathrm{d}^4x \left[\partial_t\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}\,\delta\psi\right) + \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial\nabla\psi}\,\delta\psi\right)\right] \\ &= \left[\int \mathrm{d}^3x\;\frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}\,\delta\psi\right]_{t_i}^{t_f} + \int \mathrm{d}t \oint_B \mathrm{d}^2x\,\mathbf{n}\cdot\left(\frac{\partial\mathcal{L}}{\partial\nabla\psi}\,\delta\psi\right)\,. \end{aligned} \tag{10.1.5}$$

Here $\oint_B$ denotes a surface integration over the boundary of the spatial integration region for which $\mathbf{n}$ is the outward-pointing normal vector, that arises when Gauss' theorem is used to evaluate the spatial integral over the total divergence.

If we vary over configurations that all agree at the initial and final times and on any spatial boundaries then $\delta\psi$ vanishes everywhere in space at t_i and t_f and for all time on the

[32]Having the actions for independent systems add in this way, $S = S_A + S_B$, ensures that the corresponding quantum amplitudes $e^{iS} = e^{iS_A}e^{iS_B}$ factorize, as will their squares (which in quantum mechanics gives probabilities). This is what is expected for the probabilities of statistically independent processes.

spatial boundary B. Demanding $\delta S = 0$ for arbitrary $\delta\psi$ satisfying these conditions then implies the integrand in the second term on the second line of (10.1.4) must vanish:

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_t \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} \right) - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \nabla \psi} \right) = 0 \,. \tag{10.1.6}$$

These are the classical field equations implied by the action (10.1.3).

If the action is also to be stationary for nonzero $\delta\psi$ on the boundary then we must demand that the coefficients of $\delta\psi$ also vanish on the temporal and spatial boundaries in $\delta S_{\rm s.t.}$, in addition to demanding that (10.1.6) hold everywhere else.

For instance, starting with the following simple Lagrangian that is quadratic in ψ:

$$S = \int \mathrm{d}^4 x \left[\tfrac{1}{2} \left(\partial_t \psi\right)^2 - \tfrac{1}{2} c_s^2 \nabla\psi \cdot \nabla\psi - \tfrac{1}{2} m^2 \psi^2 \right] \tag{10.1.7}$$

with constant coefficients c_s^2 and m^2 leads in this way to the field equation

$$-\partial_t^2 \psi + c_s^2 \nabla^2 \psi - m^2 \psi = 0 \,. \tag{10.1.8}$$

This is called the Klein-Gordon equation and it has plane wave solutions

$$\psi(\mathbf{r}, t) = C \, \exp\Big[-i\omega t + i \mathbf{k} \cdot \mathbf{r} \Big] \quad \text{where} \quad \omega^2 = c_s^2 \mathbf{k}^2 + m^2 \,. \tag{10.1.9}$$

When $m \to 0$ it reduces to the wave equation – *c.f.* eq. (5.3.16) – with c_s being the wave speed.

The Klein-Gordon equation is a relativistic equation when c_s is chosen to be the speed of light ($c_s \to 1$), since in that case $(\partial_t \psi)^2 - c_s^2 (\nabla\psi)^2 \to -\eta^{\mu\nu} \partial_\mu \psi \, \partial_\nu \psi$ and the combination

$$-\partial_t^2 + c_s^2 \nabla^2 \to \eta^{\mu\nu} \partial_\mu \partial_\nu = \Box \,, \tag{10.1.10}$$

becomes the d'Alembertian operator. In this case the dispersion relation for $\omega(\mathbf{k})$ has the same form as does the relativistic energy-momentum relation $E^2 = \mathbf{p}^2 + m^2$ for a particle with rest energy m. It is often encountered once fields are quantized because then its quanta are relativistic spinless particles (like pions or the Higgs boson).

10.1.2 Hamiltonian formulation

The Hamiltonian formulation of this field theory goes through straightforwardly, keeping in mind that the sum over A in quantities like $p_A \dot{q}^A$ goes over to an integral over all of space.

For the scalar field case under discussion the configuration space variables are $q^A(t) \to \psi(\mathbf{r}, t)$ and so the canonical momenta $p_A = \partial L / \partial \dot{q}^A$ become

$$\Pi(\mathbf{r}, t) := \frac{\partial \mathcal{L}}{\partial(\partial_t \psi(\mathbf{r}, t))} \,. \tag{10.1.11}$$

We are to regard this as giving Π as a functions of ψ and $\partial_t\psi$. Eq. (10.1.11) is then regarded as an expression to be solved to eliminate the time derivative $\partial_t\psi$ in favour of the ψ and Π. It is assumed here that solutions of this type to (10.1.11) exist, though we see below that this need not be so for physically interesting examples.

The system's Hamiltonian is then given by $p_A\dot{q}^A - L$, which (once the sums are turned into integrals) becomes

$$H = \int \mathrm{d}^3x \Big[\Pi\,\partial_t\psi - \mathcal{L}(\psi, \partial_t\psi)\Big] \tag{10.1.12}$$

in which $\partial_t\psi$ is traded for Π by solving (10.1.11). Once this is done the Hamiltonian is regarded as a function of ψ and Π and their spatial derivatives (but not $\partial_t\psi$). Evidently the Hamiltonian is the spatial integral over a *Hamiltonian density*, $\mathcal{H}$,

$$H = \int \mathrm{d}^3x\;\mathcal{H} \quad \text{with} \quad \mathcal{H} = \Pi\,\partial_t\psi - \mathcal{L}\,. \tag{10.1.13}$$

To see this in detail consider applying these definitions to the Klein-Gordon example, whose action is given by (10.1.7), repeated here:

$$S = \int \mathrm{d}^4x\,\left[\tfrac{1}{2}\,(\partial_t\psi)^2 - \tfrac{1}{2}c_s^2\nabla\psi\cdot\nabla\psi - \tfrac{1}{2}m^2c_s^4\psi^2\right]. \tag{10.1.14}$$

In this case (10.1.11) becomes

$$\Pi(\mathbf{r}, t) = \frac{\partial\mathcal{L}}{\partial(\partial_t\psi)} = \partial_t\psi(\mathbf{r}, t)\,, \tag{10.1.15}$$

whose solution for the 'velocity', $\partial_t\psi = \Pi$, is in this case trivial.

The Hamiltonian (10.1.12) then becomes

$$H = \int \mathrm{d}^3x\Big[\Pi^2 - \mathcal{L}(\psi, \partial_t\psi(\psi,\Pi))\Big] = \int \mathrm{d}^3x\Big[\tfrac{1}{2}\,\Pi^2 + \tfrac{1}{2}c_s^2(\nabla\psi)^2 + \tfrac{1}{2}m^2\psi^2\Big]. \tag{10.1.16}$$

This is clearly bounded from below since $\mathcal{H} \geq 0$ and it is minimized if and only if $\mathcal{H} = 0$. Because $\mathcal{H}$ is the sum of squares each of them must vanish when the energy is minimized, implying $\partial_t\psi = \nabla\psi = 0$ and $\psi = 0$. This minimum energy configuration is often called the *vacuum* configuration.

The Poisson brackets can be formulated in a similar way, keeping in mind that functions on phase space in this case are *functionals* of $\psi(\mathbf{r})$ and $\Pi(\mathbf{r})$: $F = F[\psi(\mathbf{r}), \Pi(\mathbf{r})]$. Directly applying the Poisson bracket definition (7.2.6) – and as usual converting the sum on A to an integral – gives

$$\Big\{F\,, G\Big\} = \int \mathrm{d}^3x\,\left[\frac{\delta F}{\delta\psi(\mathbf{r})}\,\frac{\delta G}{\delta\Pi(\mathbf{r})} - \frac{\delta F}{\delta\Pi(\mathbf{r})}\,\frac{\delta G}{\delta\psi(\mathbf{r})}\right]. \tag{10.1.17}$$

In particular (7.2.13) becomes

$$\Big\{\psi(\mathbf{r}, t)\,, \psi(\mathbf{r}', t)\Big\} = \Big\{\Pi(\mathbf{r}, t)\,, \Pi(\mathbf{r}', t)\Big\} = 0 \tag{10.1.18}$$

while

$$\Big\{\psi(\mathbf{r},t)\,,\psi(\mathbf{r}',t)\Big\} = \Big\{\Pi(\mathbf{r},t)\,,\Pi(\mathbf{r}',t)\Big\} = 0 \quad \text{and} \quad \Big\{\psi(\mathbf{r},t)\,,\Pi(\mathbf{r}',t)\Big\} = \delta^3(\mathbf{r}-\mathbf{r}')\,. \quad (10.1.19)$$

For some applications the Hamiltonian formulation can be less useful than the Lagrangian formulation and for others it can be preferable. In the Klein-Gordon case, for instance, the Lagrangian formulation makes Lorentz invariance manifest in the limit $c_s \to 1$ since $\mathcal{L}$ is a Lorentz scalar while $\mathcal{H}$ is not. But the Hamiltonian formulation is often more useful when quantizing fields because then the correspondence between Poisson brackets and commutators makes the commutation relations easier to guess.

Given the Hamiltonian, quantization proceeds by demanding the operators ψ and Π satisfy the canonical equal-time commutation relations suggested by the Poisson bracket relations like (10.1.19):

$$\Big[\psi(\mathbf{r},t)\,,\Pi(\mathbf{r}',t)\Big] = i\,\delta^3(\mathbf{r}-\mathbf{r}')\,. \quad (10.1.20)$$

For multiple scalar fields $\psi^a(\mathbf{r},t)$, with $a = 1,\cdots,N$, this becomes

$$\Big[\psi^a(\mathbf{r},t)\,,\Pi_b(\mathbf{r}',t)\Big] = i\,\delta^a{}_b\,\delta^3(\mathbf{x}-\mathbf{y})\,, \quad (10.1.21)$$

where $\Pi_b = \partial\mathcal{L}/\partial(\partial_t\phi^b)$ are the canonical momenta.

10.2 Electromagnetism

The next relativistic system to examine in a Lagrangian light is electromagnetism itself. In this case the Lagrangian density turns out to be simple:

$$\mathcal{L}_{EM} = \frac{1}{2}\Big(\mathbf{E}^2 - \mathbf{B}^2\Big) = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}\,, \quad (10.2.1)$$

where the first way of writing things expresses the result directly in terms of the electric and magnetic fields while the second version shows that $\mathcal{L}_{EM}$ is manifestly a Lorentz scalar, since $F_{\mu\nu}$ is an antisymmetric Lorentz tensor. In this relativistic version the indices are raised and lowered as usual using the Minkowsky metric $\eta_{\mu\nu}$ and its matrix inverse $\eta^{\mu\nu}$, so that $F_{\mu\nu}F^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\rho}F_{\mu\nu}F_{\lambda\rho} = -2\delta^{ij}F_{0i}F_{0j} + F_{ij}F^{ij} = -2\mathbf{E}^2 + 2\mathbf{B}^2$.

Although neither $\mathbf{E}$ nor $\mathbf{B}$ are differentiated in $\mathcal{L}_{EM}$, we have seen that these are not all independent of one another in any case, and the basic variables in the problem are instead the vector potential $\mathbf{A}$ and the electrostatic scalar potential Φ, related to $\mathbf{E}$ and $\mathbf{B}$ by $\mathbf{E} = -\partial_t\mathbf{A} - \nabla\Phi$ and $\mathbf{B} = \nabla\times\mathbf{A}$, or $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ where the spatial parts of A_μ are given by the components of $\mathbf{A}$ and $A^0 = -A_0 = \Phi$.

It is $\mathbf{A}$ and Φ that are to be regarded as the dynamical variables. To check this we compute δS by varying Φ and $\mathbf{A}$ to verify that this reproduces the Maxwell equations. For $S_{EM} = \int \mathrm{d}^3x\,\mathcal{L}_{EM}$ we have

$$\begin{aligned}\delta S_{EM} &= \int \mathrm{d}^4x\Big[-\mathbf{E}\cdot(\partial_t\delta\mathbf{A} + \nabla\delta\Phi) - \mathbf{B}\cdot(\nabla\times\delta\mathbf{A})\Big] \\ &= \delta S_{\mathrm{s.t.}} + \int \mathrm{d}^4x\Big[(\partial_t\mathbf{E} - \nabla\times\mathbf{B})\cdot\delta\mathbf{A} + (\nabla\cdot\mathbf{E})\,\delta\Phi\Big] \quad (10.2.2)\end{aligned}$$

where the surface term is

$$\delta S_{\rm s.t.} = -\left[\int d^3x\, \mathbf{E}\cdot\delta\mathbf{A}\right]_{t_i}^{t_f} + \int_{t_i}^{t_f} dt \oint_B d^2x\; \mathbf{n}\cdot(-\mathbf{E}\,\delta\Phi + \mathbf{B}\times\delta\mathbf{A})\,. \tag{10.2.3}$$

Requiring δS vanish for arbitrary $\delta\mathbf{A}$ and $\delta\Phi$ gives two of the (source-free) Maxwell equations

$$\nabla\cdot\mathbf{E} = 0 \quad \text{and} \quad \partial_t\mathbf{E} - \nabla\times\mathbf{B} = 0\,. \tag{10.2.4}$$

The other two Maxwell equations, $\nabla\cdot\mathbf{B} = 0$ and $\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0$, are consequences of the definitions $\mathbf{B} = \nabla\times\mathbf{A}$ and $\mathbf{E} = -\partial_t\mathbf{A} - \nabla\Phi$.

10.2.1 Constraints

Things get more interesting once we try to formulate electromagnetism using the Hamiltonian formalism. To do so the first step is to find the canonical momenta. For $\mathbf{A}$ the momentum (up to a sign) turns out to be the electric field itself, because

$$\mathbf{\Pi} = \frac{\partial\mathcal{L}_{EM}}{\partial(\partial_t\mathbf{A})} = -\mathbf{E} = \partial_t\mathbf{A} + \nabla\Phi\,. \tag{10.2.5}$$

A complication arises once we seek the momentum for Φ though because $\mathcal{L}_{EM}$ does not depend on $\partial_t\Phi$ at all, and so its canonical momentum vanishes identically:

$$\Pi_\phi = \frac{\partial\mathcal{L}_{EM}}{\partial(\partial_t\Phi)} = 0\,. \tag{10.2.6}$$

This is a constraint, and it complicates the Hamiltonian treatment of the electromagnetic field. Although (10.2.6) cannot be solved to determine $\partial_t\Phi$, this does not matter when calculating the Hamiltonian itself since $\partial_t\Phi$ only enters multiplied by Π_ϕ (which vanishes):

$$H = \int d^3x\Big[\mathbf{\Pi}\cdot\partial_t\mathbf{A} + \Pi_\phi\,\partial_t\Phi - \mathcal{L}_{EM}\Big] = \int d^3x\Big[\tfrac12(\mathbf{E}^2 + \mathbf{B}^2) + \mathbf{E}\cdot\nabla\Phi\Big]\,. \tag{10.2.7}$$

Constraints like (10.2.6) become a problem when quantizing the theory because they complicate the formulation of the Poisson brackets, whose properties are used as a guide when identifying the quantum commutation relations. It is difficult to reconcile how both $\Pi_\phi = 0$ and a canonical relation – *c.f.* eq. (7.2.13) or (10.1.19) – like $\{\Phi(\mathbf{x},t)\,,\Pi_\phi(\mathbf{y},t)\} = \delta^3(\mathbf{x}-\mathbf{y})$ can be reconciled.

It is precisely for this problem that the techniques of §11 were developed, though a full description of how it applies to electromagnetism goes beyond the scope of these notes.

11 Constrained Hamiltonian Systems

As the above chapters show, constraints arise frequently within classical mechanics, where 'contraints' here mean relations, $\phi(q,p) = 0$, that are imposed on the dynamical variables outside of their equations of motion.

In the simplest situations, such as those encountered in §1.5, the constraints express how two objects in contact move past one another (*e.g.* 'rolling without slipping' or 'sliding without friction'), and so are merely simple summaries of what are really solutions to more complicated equations of motion that express what short-range inter-atomic interactions do when two bodies come into contact. For these types of constraints §2.5 describes two ways to proceed: solve the constraints explicitly when you can and if not use the method of Lagrange multipliers.

The role of constraints takes a new turn once Hamiltonian methods are used because for these the Poisson brackets play such an important role. Constraints complicate the Poisson bracket story because it is very often true that the quantity $\phi(q, p)$ set to zero by a constraint does not have vanishing Poisson brackets with other variables, and when this happens it cannot be consistent simply to set $\phi(q, p) = 0$ (since the zero function has vanishing Poisson brackets with all other quantities). This issue becomes particularly pertinent for quantum systems, for which the Poisson bracket's role in specifying commutation relations (see §7.5) plays such a central role. Once promoted to operators should constrained variables $\phi(q, p)$ be represented by the zero operator? If so they cannot have nonzero commutation relations with other operators.

This turns out not to be just an academic exercise because *all* theories with gauge symmetries, such as electromagnetism but also including every theory that is generally coordinate invariant, are constrained systems of this type. Quantizing these theories requires knowing how to handle constrained Hamiltonian problems in a systematic way, and describing the techniques for doing so is the purpose of this chapter. The systematic formalism for handling such situations turns out to have been developed only relatively recently (during 1950s) by Dirac – yes, *that* Dirac – and that is the formalism presented here.

11.1 Constraints and Poisson Brackets

Consider, then, a system with Lagrangian $L(q, \dot{q})$, for which the equation defining momenta, $p_A = \partial L/\partial \dot{q}^A$, cannot be solved to obtain $\dot{q}^A$ as a function of the q's and p's. An obstruction for being able to do so arises whenever the Jacobian $\partial p_A/\partial \dot{q}^B$ is not invertible, which occurs whenever

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^B\, \partial \dot{q}^A}\right) = 0\,. \tag{11.1.1}$$

Primary Constraints

The condition that one or more eigenvalues of $\partial^2 L/(\partial \dot{q}^B\, \partial \dot{q}^A)$ vanish imposes a relation amongst the coordinates in the problem and is the sign that there exist relationships amongst the p's and q's of the form

$$\phi_m(q, p) \approx 0 \tag{11.1.2}$$

where the index $m = 1, \cdots, M$ counts the independent number of constraints of this type. Constraints like these that follow directly from (11.1.1) are called *primary* constraints. The symbol '≈ 0' is read as 'is *weakly* zero', where the word 'weakly' is meant to convey that although our interest is in q's and p's for which ϕ_m vanishes, the Poisson brackets of ϕ_m with other dynamical variables need not also be zero.

Although the standard construction of the Hamiltonian from the Lagrangian fails (because we cannot solve for $\dot{q}^A$ as a function of the q's and p's), for time-translation invariant systems we do expect there to exist a conserved energy, $H_c(q,p)$, that could be regarded as the Hamiltonian even though its simplest construction from the Lagrangian does not go through on the surface satisfying (11.1.2) as straightforwardly as usual. Part of the problem defining the Hamiltonian on the surface satisfying (11.1.2) is that the requirement that the Hamiltonian agree with the energy does not define it uniquely if this agreement is only required to hold when $\phi_m(q,p) = 0$, since we can always instead define it to be

$$\widetilde{H}(q,p) := H_c(q,p) + \lambda_m \phi_m(q,p)\,, \tag{11.1.3}$$

(with, as usual, an implied sum over m), for some coefficients λ_m. Any such a quantity agrees with H_c on the constrained surface because $\phi_m(q,p) \approx 0$ implies

$$\widetilde{H}(q,p) \approx H_c(q,p) \tag{11.1.4}$$

for all λ_m.

The equations of motion obtained from these different choices for Hamiltonian are not all the same, however, because (7.2.7) becomes (assuming $\partial F/\partial t = 0$)

$$\dot{F}(q,p) = \Big\{ F\,, \widetilde{H} \Big\} = \Big\{ F\,, \widetilde{H}_c \Big\} + \lambda_m \Big\{ F\,, \phi_m \Big\}\,. \tag{11.1.5}$$

In particular the equations of motion for the phase-space coordinates themselves are

$$\dot{q}^A = \Big\{ q^A\,, \widetilde{H} \Big\} = \frac{\partial H_c}{\partial p_A} + \lambda_m \frac{\partial \phi_m}{\partial p_A} \tag{11.1.6a}$$

$$\dot{p}_B = \Big\{ p_B\,, \widetilde{H} \Big\} = -\frac{\partial H_c}{\partial q^B} - \lambda_m \frac{\partial \phi_m}{\partial q^B}\,. \tag{11.1.6b}$$

These are the most general evolution equations that are consistent with the equations generated by H_c for motion along the constrained surface $\phi_m(q,p) \approx 0$.

Secondary constraints

In order for a system of constraints to be consistent we must demand that the constraints remain true as time evolves. This means that we require

$$\dot{\phi}_n(q,p) = \Big\{ \phi_n\,, \widetilde{H} \Big\} = \Big\{ \phi_n\,, H_c \Big\} + \lambda_m \Big\{ \phi_n\,, \phi_m \Big\} \approx 0\,. \tag{11.1.7}$$

If this is not already true for original collection of primary constraints then one of two things must happen. It could happen that (11.1.7) does not impose new conditions on the q's and p's and instead just implies q- and p-dependent relations amongst the λ_m's. Alternatively, the right-hand side of (11.1.7) could involve new independent functions of the q's and p's and if so then these are new constraints – called *secondary* constraints – that must also weakly vanish if the constraints are to be consistent with time evolution.

In the latter case these new constraints must be added to the primary constraints and (11.1.7) computed again, possibly introducing further secondary constraints, with the process repeated until no new constraints are generated by time evolution. Once this is done the complete list of constraints is longer

$$\phi_a(q,p) = 0 \quad \text{with } a = 1, \cdots, T \tag{11.1.8}$$

where M is the number of primary constraints appearing in (11.1.2) and $T = M + L$ where L is the number of new secondary constraints generated by time evolution.

As mentioned above, consistency with the equations of motion also imposes relations amongst the λ_a's, and these imply that the coefficients λ_a can be solved to give $\lambda_a = \lambda_a(q,p)$. This is what allows the Hamiltonian to be expressed as $\widetilde{H} = \widetilde{H}(q,p)$.

11.1.1 Classes of Constraints

Given a self-consistent set of constraints like (11.1.8), then functions on phase space can be usefully divided into *First Class* and *Second Class* according to whether or not their Poisson bracket with the constraints are all weakly zero. That is, by definition $R(q,p)$ is a first-class function on phase space if

$$\Big\{ R \,, \phi_a \Big\} \approx 0 \;\forall a = 1, \cdots, T \qquad \text{(first-class function)}\,, \tag{11.1.9}$$

A second-class function on phase space is one that is not first class: it has a nonzero Poisson bracket with at least one of the constraints in (11.1.8). Clearly the definition of second-class functions is ambiguous because one can always add to it a first-class function without changing is Poisson brackets with the constraints.

The collection of constraints themselves can in particular be separated into first-class and second-class categories because the ϕ_n's are themselves examples of functions on phase space. Once this is done it is also true that the square of a second-class constraint is itself a first-class constraint because the definition of the Poisson bracket implies

$$\Big\{ \phi_n^2 \,, \phi_a \Big\} = 2\phi_n \Big\{ \phi_n \,, \phi_a \Big\} \approx 0 \,, \tag{11.1.10}$$

where the final weak equality follows because all of the constraints satisfy $\phi_a \approx 0$.

Once sorted into first-class and second-class constraints, the constraints are denoted $\{\phi_a\} = \{\psi_i \,, \varphi_\alpha\}$ where ψ_i (for $i = 1, \cdots, I$) are the first-class constraints and φ_α (for $\alpha =$

$1, \cdots, N$) are the second-class constraints. Here $I + N = T$, where recall that T counts the number of primary and secondary constraints: $a = 1, \cdots, T$. Notice that the original primary constraints can be either first class or second class, and the same is also true for the secondary constraints.

The distinction between first-class and second-class constraints is the crucial one. As we see below, second-class constraints are just ordinary constraints in the way that was encountered in the more elementary treatments of §1.5 or §2.5. The way they will be handled parallels what was done for these constraints in earlier sections. The first-class constraints are different though. As mentioned above, the freedom to add one of these to any second-class constraint introduces a fundamental ambiguity into the definition of second-class constraints. In this sense their presence is closely related to the existence of *gauge symmetries*, which are ambiguiities in the definitions of variables, the most famous of which is the ambiguity $\mathbf{A} \to \mathbf{A} + \nabla\chi$ and $\Phi \to \Phi - \partial_t\chi$ that leaves physical quantities – *i.e.* the electric field $\mathbf{E} = -\partial_t\mathbf{A} - \nabla\Phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$ – completely unchanged. The strategy for dealing with first-class constraints is similar to what one does with gauge symmetries: introduce new constraints ('choose a gauge') that convert the first-class constraints into second-class ones. Then use the same formalism that is used for any other type of second-class constraints.

11.1.2 Dirac Brackets

An important property of the second-class constraints is that the matrix $C_{\alpha\beta}$, defined by

$$C_{\alpha\beta} := \Big\{\varphi_\alpha \,, \varphi_\beta\Big\} \tag{11.1.11}$$

is invertible. If it were not invertible it would have a zero eigenvector, but this eigenvector is then a first-class constraint because the linear combination of the φ_α's to which it corresponds by construction has a vanishing Poisson bracket with all of the other constraints. So once all first-class constraints are identified the remaining second class constraints produce a matrix $C_{\alpha\beta}$ that can be inverted. But its definition also implies $C_{\alpha\beta} = -C_{\beta\alpha}$ is antisymmetric, and any odd-dimensional antisymmetric matrix has a zero eigenvector. So there must always be an even number of second-class constraints.

Denoting the inverse matrix for $C_{\alpha\beta}$ by $C^{-1}_{\alpha\beta}$, we know

$$C_{\alpha\beta}C^{-1}_{\beta\gamma} = C^{-1}_{\alpha\beta}C_{\beta\gamma} = \delta_{\alpha\gamma}\,, \tag{11.1.12}$$

because these are the component versions of the matrix equations $CC^{-1} = C^{-1}C = 1$. What is useful about C^{-1} is that it allows us to take any phase-space quantity $F(q,p)$ and build from it a new quantity $\widehat{F}(q,p)$ that is equal to $F(q,p)$ when the constraints are satisfied but whose Poisson bracket with all of the second-class constraints is weakly zero.

To this end define

$$\widehat{F}(q,p) := F(q,p) - \Big\{F \,, \varphi_\alpha\Big\} C_{\alpha\beta}\, \varphi_\beta\,. \tag{11.1.13}$$

Clearly this satisfies

$$\begin{aligned}\Big\{\widehat F\,,\varphi_\gamma\Big\} &\approx \Big\{F\,,\varphi_\gamma\Big\} - \Big\{F\,,\varphi_\alpha\Big\}C_{\alpha\beta}\Big\{\varphi_\beta\,,\varphi_\gamma\Big\}\\ &= \Big\{F\,,\varphi_\gamma\Big\} - \Big\{F\,,\varphi_\alpha\Big\}C_{\alpha\beta}\,C_{\beta\gamma} = \Big\{F\,,\varphi_\gamma\Big\} - \Big\{F\,,\varphi_\gamma\Big\} = 0\,, \end{aligned} \tag{11.1.14}$$

as claimed, where the first line follows because only the Poisson bracket between φ_β and φ_γ must be computed in the second term because all other contributions are explicitly proportional to φ_β and so are weakly zero. Notice that there is no guarantee that $\{\widehat F\,,\psi_i\}$ is weakly zero for first-class constraints.

Because the Poisson bracket of $\widehat F(q,p)$ with second-hand constraints is weakly zero, it follows that there is no inconsistency between its Poisson brackets and choosing $\varphi_\alpha \approx 0$. For instance if the rule for quantizing is to replace $\widehat F$ with an operator $F_{\rm op}$ (as opposed to replacing F itself with $F_{\rm op}$), with the Poisson bracket going over to the commutator, then there is nothing inconsistent about the constraints $\varphi_\alpha \simeq 0$ being represented by the zero operator (with which all other operators must commute), because $\{\widehat F\,,\varphi_\alpha\} \simeq 0$ for all φ_α and all F.

We see from this that all second-class constraints can be made consistent with the Poisson bracket structure simply by replacing every phase-space observable $F(q,p)$ by $\widehat F(q,p)$, defined by (11.1.13). Notice also that although $\widehat F \approx F$ – so F and $\widehat F$ agree with one another once the constraints are satisfied – it is *not* true that $\{\widehat F\,,\widehat G\} \approx \{F\,,G\}$.

It *is* possible to define a new bracket $\{F\,,G\}_D$ that has the property that $\{\widehat F\,,\widehat G\} \approx \{F\,,G\}_D$, however. The required bracket is called the *Dirac bracket* and is defined by

$$\Big\{F\,,G\Big\}_D := \Big\{F\,,G\Big\} - \Big\{F\,,\varphi_\alpha\Big\}\,C^{-1}_{\alpha\beta}\,\Big\{\varphi_\beta\,,G\Big\}\,. \tag{11.1.15}$$

Straightforward application of the definitions implies the Dirac bracket satisfies

$$\Big\{F\,,G\Big\}_D \approx \Big\{\widehat F\,,\widehat G\Big\} \approx \Big\{\widehat F\,,G\Big\} \approx \Big\{F\,,\widehat G\Big\}\,, \tag{11.1.16}$$

as well as the Jacobi identity

$$\Big\{A\,,\Big\{B\,,C\Big\}_D\Big\}_D + \Big\{B\,,\Big\{C\,,A\Big\}_D\Big\}_D + \Big\{C\,,\Big\{A\,,B\Big\}_D\Big\}_D \approx 0\,. \tag{11.1.17}$$

As applied to the Hamiltonian H_c the above procedure leads to

$$\widehat H := H_c - \Big\{H_c\,,\varphi_\alpha\Big\}\,C^{-1}_{\alpha\beta}\,\varphi_\beta\,, \tag{11.1.18}$$

and so comparing this to (11.1.3) shows that

$$\widetilde H = \widehat H \quad \text{with} \quad \lambda_\beta = -\Big\{H_c\,,\varphi_\alpha\Big\}\,C^{-1}_{\alpha\beta}\,. \tag{11.1.19}$$

11.2 Applications*

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11.2.1 Relativistic Point Particle (again)

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11.2.2 Electromagnetism (again)

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A Refreshers on useful tools

This appendix gathers together some topics that are normally seen in a physics program but which might need refreshing for this course.

A.1 Calculus of Variations

The calculus of variations is that branch of mathematics that works out how to do calculus – *i.e.* differentiate and integrate – for *functionals* rather than functions. Recall that a function can be regarded as a map between two sets $f : A \to B$ having the property that for each of its arguments, $a \in A$, the function returns a unique $b = f(a) \in B$. The set A is called the function's *domain* and the set B is called its *range.* For a real function the range $B = \mathbb{R}$ is the set of real numbers. For 'ordinary' functions, $x(t)$, of everyday experience the domain is also the real numbers (or a subset thereof).

From this perspective a functional is just a function whose argument is itself a function (or a set of real functions) or a 'path', $x(t)$, that maps $\mathbb{R} \to \mathbb{R}$. A real functional is a rule that returns a real number for each path in its domain. For the functionals encountered in physics the rule defining the functional often arises as an integral over the specified path, such as the length, $s[\mathbf{x}(t)]$, of a path in n-dimensional flat Euclidean space, E_n.

For instance, suppose we consider the set of paths $\mathbf{x}(t)$ where the parameter t along the path runs from $t_0 \leq t \leq t_1$. Then the tangent vector to the path $\mathbf{x}(t)$ is given by $\dot{\mathbf{x}} = \mathrm{d}\mathbf{x}/\mathrm{d}t$ and the length of the path is the functional

$$s[\mathbf{x}(t)] = \int_{t_0}^{t_1} \mathrm{d}t \; |\dot{\mathbf{x}}(t)| = \int_0^1 \mathrm{d}t \; \sqrt{\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}} = \int_0^1 \mathrm{d}t \; \sqrt{\dot{x}_1^2(t) + \dot{x}_2^2(t) + \cdots + \dot{x}_n^2(t)}\,. \tag{A.1.1}$$

Notice that the integral is invariant under changes in how the curve $\mathbf{x}(t)$ is parameterized because changing to u where $t = t(u)$ is a monotonic function because the Jacobian $|\mathrm{d}t/\mathrm{d}u|$ cancels between the integration measure $\mathrm{d}t = \mathrm{d}u\,|\mathrm{d}t/\mathrm{d}u|$ and the change to the integrand $\sqrt{(\mathrm{d}\mathbf{x}/\mathrm{d}t)^2} = |\mathrm{d}u/\mathrm{d}t|\sqrt{(\mathrm{d}\mathbf{x}/\mathrm{d}u)^2}$. As a result we are free to parameterize the curve any way we like when computing $s[\mathbf{x}(t)]$ and we use this freedom to arrange that the initial and final points on the curve are labelled by $t_0 = 0$ and $t_1 = 1$ respectively.

The calculus of variations enters when we ask optimization questions like: which path between to fixed points has the minimum length? To answer these types of questions it is useful to have a notion of differentiation for functionals, in order to use the criterion that the first derivative of a functional must vanish when evaluated at a minimum (or maximum or saddle point).

The rule for defining derivatives of functionals is designed to follow what one does for derivatives of ordinary functions. For functions the derivative is defined as a limiting process, where one computes the slope of the straight line that connects $x(t)$ and $x(t + \Delta t)$ in the limit that Δt is taken to zero:

$$\dot{x}(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} \,. \tag{A.1.2}$$

By analogy the same thing is done for a functional like $s[x(t)]$: one first compares the value, $s[x(t) + \delta x(t)]$ for a path $x(t) + \delta x(t)$ that differs slightly from the initial path, $x(t)$, and computes the difference: $\delta s := s[x(t) + \delta x(t)] - s[x(t)]$. By contruction, this must vanish when $\delta x(t) = 0$ and so for small enough $\delta x(t)$ it should be sufficient to Taylor expand in powers of $\delta x(t)$ and stop at linear order. One then defines the coefficient of $\delta x(t)$ to be the derivative, denoted $\delta s/\delta x(t)$:

$$\delta s := s[x(t) + \delta x(t)] - s[x(t)] = \int_0^1 \mathrm{d}t \, \left(\frac{\delta s}{\delta x(t)} \right) \delta x(t) + \mathcal{O}[(\delta x)^2] \,. \tag{A.1.3}$$

When varying the path there is a choice whether or not one is free to vary the enpoints of the path $x(0)$ or $x(1)$ as well as the points in between. The simplest case chooses not to vary the endpoints, in which case we consider only variations $\delta x(t)$ that vanish at the endpoints: $\delta x(0) = \delta x(1) = 0$. The results found in this case also apply when the endpoints are allowed to be varied because having the freedom to vary the endpoints includes the case where one chooses not to vary the endpoints. Varying the endpoints introduces new conditions at the endpoints (as described later in this section).

To see how this works consider the example where $s[\mathbf{x}(t)] = s[x_1(t), \dots, x_n(t)]$ is the length of a path in n-dimensional Euclidean space, as defined in (A.1.1). In this case applying the definition (A.1.3) gives

$$\begin{aligned} \delta s &= \int_0^1 \mathrm{d}t \, \left\{ \sqrt{(\dot{x}_1 + \delta\dot{x}_1)^2 + \cdots + (\dot{x}_n + \delta\dot{x}_n)^2} - \sqrt{\dot{x}_1^2 + \cdots + \dot{x}_n^2} \right\} \\ &= \int_0^1 \mathrm{d}t \, \left\{ \frac{\dot{x}_1 \, \delta\dot{x}_1 + \cdots + \dot{x}_n \, \delta\dot{x}_n}{\sqrt{\dot{x}_1^2 + \cdots + \dot{x}_n^2}} + \mathcal{O}\Big(\delta x_i^2\Big) \right\} \,, \end{aligned} \tag{A.1.4}$$

where we use that the integrand is an ordinary function and we already know how to differentiate and Taylor expand those. Eq. (A.1.4) does not quite have the form sought on the right-hand side of (A.1.3) because in (A.1.4) the deviation $\delta x_i(t)$ appears differentiated. To fix this we integrate by parts to remove the derivative from δx_i, leading to

$$\begin{aligned} \delta s = &\left[\frac{\dot{x}_1 \, \delta x_1 + \cdots + \dot{x}_n \, \delta x_n}{\sqrt{\dot{x}_1^2 + \cdots + \dot{x}_n^2}} \right]_{t=0}^{t=1} \\ &- \int_0^1 \mathrm{d}t \, \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\dot{x}_1}{\sqrt{\dot{x}_1^2 + \cdots + \dot{x}_n^2}} \right] \delta x_1 + \cdots + \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\dot{x}_n}{\sqrt{\dot{x}_1^2 + \cdots + \dot{x}_n^2}} \right] \delta x_n + \mathcal{O}\Big(\delta x_i^2\Big) \right\} \,. \end{aligned} \tag{A.1.5}$$

If we restrict to variations that do not change the endpoints then we know $\delta x_i(0)$ for all i and so the surface term on the first line of (A.1.5) vanishes. What remains on the right-hand side has the same form as (A.1.3) and so we can read off the functional derivatives:

$$\frac{\delta s}{\delta x_i(t)} = -\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\dot{x}_i}{\sqrt{\dot{x}_1^2 + \cdots + \dot{x}_n^2}}\right] \qquad (\text{for all } 0 < t < 1 \text{ and each } i = 1, \cdots, n)\,. \tag{A.1.6}$$

Notice that the resulting functional derivatives are themselves functions of t. We have an independent derivative for each independent label for $x_i(t)$; that is, a separate derivative for each i and t.

We can now use this to see which curves minimize the path length between two fixed points, by asking which curves make $\delta s/\delta x_i(t)$ vanish for all i and for all $0 < t < 1$. Although the n equations found by setting (A.1.6) to zero look complicated they have a very simple solution: $\dot{x}_i$ is a constant for all i. The paths with shortest lengths therefore are linear functions:

$$x_i(t) = A_i + B_i t \tag{A.1.7}$$

where A_i and B_i are integration constants. These integration constants are found by requiring the solution to pass through the two specified endpoints, $x_i(0) = a_i$ and $x_i(1) = b_i$, and so

$$A_i = a_i \quad \text{and} \quad B_i = b_i - a_i\,. \tag{A.1.8}$$

Because the tangents to the curves defined by (A.1.7) are constant vectors these curves can be recognized as the equations for straight lines. Famously, in Euclidean geometry the shortest distance between to points is a straight line.

A.1.1 Varying the endpoints

We can also see now what happens if we allow ourselves the freedom to vary the endpoints in addition to varying the paths in between the endpoints. Demanding δs vanish for *all* variations $\delta x_i(t)$ includes as a special case the situation where we do not vary the endpoints. As a consequence we still can concluded that $\delta s/\delta x_i$ as given in (A.1.6) must vanish for each $i = 1, \cdots, n$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\dot{x}_i}{\sqrt{\dot{x}_1^2 + \cdots + \dot{x}_n^2}}\right] \qquad (\text{for } 0 < t < 1 \text{ and each } i = 1, \cdots, n)\,. \tag{A.1.9}$$

This in turn ensures that the second line of (A.1.5) vanishes.

But the first line must also vanish if we are to successfully have extremized s and because δx_i need *not* vanish at the endpoints this requires we must also demand separately that the surface term on the first line of (A.1.5) should vanish. Because the variations at each end are independent this means that

$$\frac{\dot{x}_1\,\delta x_1 + \cdots + \dot{x}_n\,\delta x_n}{\sqrt{\dot{x}_1^2 + \cdots + \dot{x}_n^2}} = 0\,, \tag{A.1.10}$$

separately, for both $t = 0$ and $t = 1$. Because δx_i need *not* vanish there anymore, eq. (A.1.10) requires our solutions to (A.1.9) must also satisfy the boundary conditions

$$\dot{x}_i(0) = \dot{x}_i(1) = 0 \qquad \text{(for all } i\text{)}\,. \tag{A.1.11}$$

Not surprisingly, if we ask for the curve with the shortest distance and if we are also allowed to move the endpoints to help achieve this then the shortest curve has $x_i(t)$ independent of t and so the two endpoints of the curve lie at the same point: the shortest curve in this case has zero length.

A.2 Method of Lagrange multipliers

Suppose one wishes to minimize or maximize a function $h(x, y)$ of two independent variables, x and y. This is relatively easy to do, by finding those $x = x_m$ and $y = y_m$ that make the partial derivatives of h vanish:

$$\left(\frac{\partial h}{\partial x}\right)_{x_m, y_m} = \left(\frac{\partial h}{\partial y}\right)_{x_m, y_m} = 0\,. \tag{A.2.1}$$

For instance suppose x and y locally describe local coordinates for the surface of the earth and $h(x, y)$ describes the local height of the terrain for some part of the earth's surface. Then x_m and y_m might pick out the highest and lowest points of the terrain in the local area.

A more complicated problem is to minimize h subject to a constraint that relates x and y. For instance, suppose a road crosses the terrain with a route described by the curve $c(x, y) = 0$ for some function $c(x, y)$. Then a constrained problem might ask for the highest or lowest point of the terrain that is encountered by someone who moves only along this road. This is not necessarily found by (A.2.1) unless x_m and y_m happen by chance also to satisfy $c(x_m, y_m) = 0$.

In principle this constrained problem can be handled by solving the constraint $c(x, y) = 0$ to find the function $y = \mathfrak{y}(x)$ that satisfies $c(x, \mathfrak{y}(x)) \equiv 0$ as an identity for all x. For instance if $c(x, y) = x^2 + y^2 - 1$ then the constraint $c(x, y) = 0$ implies $x^2 + y^2 = 1$, a condition solved by $y = \mathfrak{y}(x) = \pm\sqrt{1 - x^2}$. Once this constraint is solved then evaluating $h(x, y)$ along this curve gives a function of the single independent variable x: $\mathfrak{h}(x) := h(x, \mathfrak{y}(x))$ that expresses the values for $h(x, y)$ seen along the curve. The minima of $\mathfrak{h}(x)$ are then found (as usual) by setting to zero the derivative of h restricted to lie along this curve:

$$\left(\frac{\mathrm{d}\mathfrak{h}(x)}{\mathrm{d}x}\right)_{x_c} = \left(\frac{\partial h}{\partial x}\right)_{x_c, \mathfrak{y}(x_c)} + \left(\frac{\mathrm{d}\mathfrak{y}}{\mathrm{d}x}\right)_{x_c} \left(\frac{\partial h}{\partial y}\right)_{x_c, \mathfrak{y}(x_c)} = 0\,. \tag{A.2.2}$$

A variation on this approach chooses a more general parameterization of the constraint, by finding functions $x(\theta)$ and $y(\theta)$ that satisfy $c(x(\theta), y(\theta)) = 0$ for all θ. For instance, the choices $x(\theta) = \cos\theta$ and $y(\theta) = \sin\theta$ have this property for the example $c(x, y) = x^2 + y^2 - 1$. In this case the extrema are found by differentiating $h(x(\theta), y(\theta))$ with respect to θ and setting the first derivative to zero. These types of alternative parameterizations can be useful because they can sometimes avoid the singular behaviour seen above in $\mathfrak{h}(x)$ near $x = \pm 1$ (which are points where the parameterization of the curve in terms of x break down.

The method of lagrange multipliers is just a simpler way to obtain x_c without having to explicitly solve for the curve $y = \mathfrak{y}(x)$. The idea is instead to minimize the quantity

$$f(x, y, \lambda) := h(x, y) + \lambda\, c(x, y) \tag{A.2.3}$$

with respect to all three independent variables x, y and λ. This leads to the following three equations:

$$\frac{\partial f}{\partial x} = \frac{\partial h}{\partial x} + \lambda \frac{\partial c}{\partial x} = 0\,, \quad \frac{\partial f}{\partial y} = \frac{\partial h}{\partial y} + \lambda \frac{\partial c}{\partial y} = 0 \quad \text{and} \quad \frac{\partial f}{\partial \lambda} = c = 0\,, \tag{A.2.4}$$

whose solutions can be denoted x_l, y_l and λ_l. Lagrange's claim is that (x_l, y_l) coincide with (x_c, y_c) and so the solutions to (A.2.4) agree with those of (A.2.2).

To see why this is so first notice that the third equation in (A.2.4) states that $c(x_l, y_l) = 0$ and so $y_l = \mathfrak{y}(x_l)$ lies on the constraint curve. Next, notice that the derivative of $\mathfrak{y}(x)$ is related to those of $c(x, y)$, using the fact that $c(x, \mathfrak{y}(x)) \equiv 0$ is an identity for all x, and so remains true once differentiated. Differentiating this with respect to x using the chain rule leads to the conclusion (also true for all x)

$$\frac{\partial c}{\partial x} + \frac{\partial c}{\partial y}\frac{\mathrm{d}\mathfrak{y}}{\mathrm{d}x} = 0 \quad \text{and so} \quad \left(\frac{\mathrm{d}\mathfrak{y}}{\mathrm{d}x}\right)_{x_l} = -\left(\frac{\partial c/\partial x}{\partial c/\partial y}\right)_{x_l, y_l}. \tag{A.2.5}$$

Finally taking the following linear combination of the first two equations of (A.2.4) and using (A.2.5) leads to the inference

$$\begin{aligned} 0 &= \left(\frac{\partial f}{\partial x}\right)_{x_l, y_l, \lambda_l} + \left(\frac{\mathrm{d}\mathfrak{y}}{\mathrm{d}x}\right)_{x_l} \left(\frac{\partial f}{\partial y}\right)_{x_l, y_l, \lambda_l} \\ &= \left(\frac{\partial f}{\partial x}\right)_{x_l, y_l, \lambda_l} - \left(\frac{\partial c/\partial x}{\partial c/\partial y}\right)_{x_l, y_l, \lambda_l} \left(\frac{\partial f}{\partial y}\right)_{x_l, y_l, \lambda_l} \\ &= \left(\frac{\partial h}{\partial x} + \lambda \frac{\partial c}{\partial x}\right)_{x_l, y_l, \lambda_l} - \left(\frac{\partial c/\partial x}{\partial c/\partial y}\right)_{x_l, y_l, \lambda_l} \left(\frac{\partial h}{\partial y} + \lambda \frac{\partial c}{\partial y}\right)_{x_l, y_l, \lambda_l} \\ &= \left(\frac{\partial h}{\partial x} + \frac{\mathrm{d}\mathfrak{y}}{\mathrm{d}x}\frac{\partial h}{\partial y}\right)_{x_l, y_l, \lambda_l}, \end{aligned} \tag{A.2.6}$$

which shows that (x_l, y_l) automatically satisfy (A.2.2) and so $(x_l, y_l) = (x_c, y_c)$, as claimed.

A.3 Vector identities and Kronecker and Levi-Civita tensors

This appendix derives several of the vector identities that arise frequently in the main text. These identities ultimately trace their roots to the properties of 3×3 rotation matrices, R.

That is, we saw in the main text that under a rotation of basis vectors the components of a given vector $\mathbf{V}$ get transformed by matrix multiplication, $\mathbf{V} \to R\mathbf{V}$, or in components, $V_i \to \sum_j R_{ij} V_j$ (see eq. (1.6.10)). Here the rotation matrices R are 3-dimensional orthogonal matrices – *i.e.* those satisfying $R^T R = I$. In component form the orthogonality condition becomes

$$\sum_j R_{ji} R_{jk} = \delta_{ik} \,, \tag{A.3.1}$$

where the Kronecker delta $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$ are the components of the unit matrix.

A.3.1 Invariant tensors

The set of orthogonal $N \times N$ matrices forms a group, denoted $O(N)$, and so rotations in 3 dimensions involve the group $O(3)$. Notice that taking the determinant of the orthogonality condition implies

$$\det(R^T R) = (\det R)^2 = 1 \,, \tag{A.3.2}$$

which uses the properties $\det(AB) = (\det A)(\det B)$ and $\det I = 1$. It follows that $\det R = \pm 1$. The subgroup of $O(3)$ with unit determinant is called $SO(3)$ (for special orthogonal group). The matrix $-I$ is an example of an element of $O(3)$ that is not in $SO(3)$.

The components of a matrix M similarly transform under rotations by undergoing a similarity transformation: $M \to RMR^T$, or in components

$$M_{ij} \to \sum_{kl} R_{ik} M_{kl} R_{jl} \,. \tag{A.3.3}$$

These rules ensure that a matrix product like $M\mathbf{V}$ transforms the same way as does $\mathbf{V}$ because $M\mathbf{V} \to (RMR^T)(R\mathbf{V}) = R(M\mathbf{V})$.

From this point of view the unit matrix is special: it is the only 3×3 matrix that is completely unchanged by rotations because orthogonality ensures $RIR^T = RR^T = I$. In that sense the Kronecker delta can be regarded as being an *invariant tensor*. Substituting $M_{ij} = \delta_{ij}$ into the right-hand side of (A.3.3) returns δ_{ij}, same as on the left-hand side, by virtue of (A.3.1).

The δ_{ij} is the only invariant tensor for $O(3)$ but there is a second tensor that is almost an invariant tensor. This is the Levi-Civita tensor ϵ_{ijk}, which is defined to be completely antisymmetric under the interchange of any pair of indices, so ϵ_{ijk} must vanish if any two indices take the same value (*e.g.* $\epsilon_{112} = 0$). The value when all indices are different is uniquely fixed by choosing the convention that $\epsilon_{123} = +1$ and all others are found by permutation (so *e.g.* $\epsilon_{321} = -\epsilon_{123} = -1$ and so on).

Notice in particular that *any* 3-index tensor, A_{ijk}, that is completely antisymmetric under the exchange of any pair of indices must be proportional to ϵ_{ijk}. In particular, consider evaluating the following quantity for any matrix M_{ij}:

$$A_{ijk} := \sum_{lmn} \epsilon_{lmn} M_{il} M_{jm} M_{kn} \,. \tag{A.3.4}$$

Because this is completely antisymmetric under the interchange of any two indices it must be proportional to ϵ_{ijk}. Explicit evaluation for a particular matrix M shows that the proportionality constant is the determinant: $\det M$, so

$$\sum_{lmn} \epsilon_{lmn} M_{il} M_{jm} M_{kn} = (\det M) \epsilon_{ijk} \,. \tag{A.3.5}$$

Now comes the main point: specializing (A.3.5) to $M = R$ where R is an orthogonal matrix shows that

$$\sum_{lmn} \epsilon_{lmn} R_{il} R_{jm} R_{kn} = \pm \epsilon_{ijk} \,, \tag{A.3.6}$$

where the sign is the determinant of R (which must be a sign because of (A.3.2)). This shows that the Levi-Civita tensor is an invariant tensor of the group $SO(3)$ but is only an invariant *pseudotensor* under $O(3)$ because it changes sign when $\det R = -1$.

But if ϵ_{ijk} at most changes sign under an arbitrary $O(3)$ transformation then it must be that any quantity that is quadratic in the Levi-Civita tensor must be an invariant tensor of $O(3)$ and so must be constructable in terms of the Kronecker delta. This can be verified explicitly, leading to the identity

$$\epsilon_{ijk}\, \epsilon_{abc} = \delta_{ia}\, \delta_{jb}\, \delta_{kc} \pm (5 \text{ other permutations}) \tag{A.3.7}$$

where δ_{ij} is the usual Kronecker delta that is unity if $i = j$ and zero otherwise. This last identity can be proven by verifying that both sides agree on the symmetry of their indices and checking specific values for a, b, c, i, j, k. A second pair of identities comes from contracting the above with Kronecker deltas:

$$\epsilon_{ijk}\, \epsilon_{abc}\, \delta^{ia} = \delta_{jb}\, \delta_{kc} - \delta_{jc}\, \delta_{kb} \quad \text{and} \quad \epsilon_{ijk}\, \epsilon_{abc}\, \delta^{ia}\, \delta^{jb} = 2\, \delta_{kc} \tag{A.3.8}$$

and so $\epsilon_{ijk} \epsilon^{ijk} = 6$.

A.3.2 Vector identities

Why do we care? Invariant tensors are useful because they appear when we multiply vectors, using the dot or cross product. In components the dot product of two vectors can be written using the Kronecker tensor

$$c = \mathbf{a} \cdot \mathbf{b} = \sum_{ij} a_i b_j \delta_{ij} = a_1 b_1 + a_2 b_2 + a_3 b_3 \,. \tag{A.3.9}$$

In components the cross product of two vectors $\mathbf{d} = \mathbf{a} \times \mathbf{b}$ can be written in terms of the Levi-Civita tensor as

$$d_i = \sum_{jk} \epsilon_{ijk} a_j b_k \,, \tag{A.3.10}$$

and so

$$d_1 = a_2 b_3 - a_3 b_2 \,, \quad d_2 = a_3 b_1 - a_1 b_3 \,, \quad d_3 = a_1 b_2 - a_2 b_1 \,. \tag{A.3.11}$$

Vector identities arise when products involving these quantities are simplified using the properties of ϵ_{ijk} or identities like (A.3.7) or its children (A.3.8). For example antisymmetry of ϵ_{ijk} together with $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \epsilon_{ijk} a_i b_j c_k$ implies

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \,. \tag{A.3.12}$$

Similarly, the result of two cross products: $\mathbf{d} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ has components

$$d_i = \sum_{jklm} \epsilon_{ijk} a_j (\epsilon_{klm} b_l c_m) = \sum_{jlm} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_j b_l c_m = \sum_j (a_j b_i c_j - a_j b_j c_i) \tag{A.3.13}$$

where the second equality uses the first of eqs. (A.3.8). Consequently for any three vectors we have the identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \,. \tag{A.3.14}$$

Another useful identity of this type gives the dot product of two cross products:

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= \sum_{ijkmn} (\epsilon_{ijk} a_j b_k)(\epsilon_{imn} c_m d_n) = \sum_{jkmn} (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) a_j b_k c_m d_n \\ &= \sum_{jk} (a_j b_k c_j d_k - a_j b_k c_k d_j) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \,. \end{aligned} \tag{A.3.15}$$

A.4 Vector calculus

This appendix contains a (very short) review of the basic facts of multivariate vector calculus (in three spatial dimensions) used in the main text.

The gradient of a scalar field is perhaps the simplest vector derivative to define. Consider a scalar field $\phi(\mathbf{x})$ that returns a real number for each position throughout space. (Temperature as a function of position is an example of such a scalar field.) A vector field can be built from its derivatives:

$$\nabla\phi := \frac{\partial\phi}{\partial x}\,\mathbf{e}_x + \frac{\partial\phi}{\partial y}\,\mathbf{e}_y + \frac{\partial\phi}{\partial z}\,\mathbf{e}_z = \partial_i\phi\,\mathbf{e}_i \,, \tag{A.4.1}$$

where there is an implied sum over i in the last form, $\{x, y, z\}$ are the three Cartesian coordinates and $\mathbf{e}_i$ point along the three Cartesian axes so that $\mathbf{x} = x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z$. This transforms under rotation the way the notation suggests: it is a 3-dimensional vector. Geometrically, the vector $\nabla\phi(\mathbf{x})$ points in the direction along which ϕ increases the most quickly starting at the given point $\mathbf{x}$.

A vector field,

$$\mathbf{A}(\mathbf{x}) = A_x(\mathbf{x})\,\mathbf{e}_x + A_y(\mathbf{x})\,\mathbf{e}_y + A_z(\mathbf{x})\,\mathbf{e}_z = A_i\,\mathbf{e}_i\,, \tag{A.4.2}$$

is a vector that is specified independently at each position throughout space. For physical applications its component functions are usually imagined to be smooth enough that it can be differentiated as many times as needed.

There is more than one way to combine derivatives with vector fields in a way that transforms sensibly (*i.e.* as a tensor) under rotations. In particular given a vector field $\mathbf{A}(\mathbf{x})$ one can always define a scalar field by taking the 'divergence':

$$\operatorname{div}\mathbf{A} := \nabla\cdot\mathbf{A} := \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \partial_i A_i \tag{A.4.3}$$

This is a scalar in the sense that if $\{x', y', z'\}$ are rotations of $\{x, y, z\}$ then the formula for $\nabla\cdot\mathbf{A}$ is precisely the same as above, but using the primed coordinates: $\nabla\cdot\mathbf{A} = \partial_{x'}A_{x'} + \partial_{y'}A_{y'} + \partial_{z'}A_{z'}$.

A second combination of derivatives – the 'curl' – of a vector field transforms as a vector, and is defined by

$$\begin{aligned}\operatorname{curl}\mathbf{A} := \nabla\times\mathbf{A} &= \left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right)\mathbf{e}_x + \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\mathbf{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{e}_z \\ &= \mathbf{e}_i\,\epsilon_{ijk}\partial_j A_k\,,\end{aligned} \tag{A.4.4}$$

where ϵ_{ijk} is the completely antisymmetric Levi-Civita symbol discussed in the previous Appendix.

A straightforward application of the definitions shows that the following two identities are true for any multiply differentiable scalar and vector field. The symmetry of a second derivative — *e.g.* $\partial^2\phi/\partial x\partial y = \partial^2\phi/\partial y\partial x$ — implies the curl of a gradient vanishes:

$$\nabla\times(\nabla\phi) = 0\,, \tag{A.4.5}$$

for any $\phi(\mathbf{x})$. The inverse of this is also locally true: if a vector $\mathbf{A}$ satisfies $\nabla\times\mathbf{A} = 0$ in some region around a point $\mathbf{x}$ then it is also true that there exists a scalar ϕ such that $\mathbf{A} = \nabla\phi$, at least in a sufficiently small region around $\mathbf{x}$.

A similar statement for the second derivative of a vector field states that the divergence of a curl vanishes:

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0\,, \tag{A.4.6}$$

for any $\mathbf{A}(\mathbf{x})$. Besides being sufficient this is also (locally) necessary: if a vector field satisfies $\nabla\cdot\mathbf{A} = 0$ in some region about a point $\mathbf{x}$ then there exists another vector field $\mathbf{C}$ such that $\mathbf{A} = \nabla\times\mathbf{C}$, at least in some sufficiently small region around $\mathbf{x}$.

Applying a divergence to a gradient similarly gives the Laplace operator:

$$\nabla \cdot (\nabla \phi) = \nabla^2 \phi := \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\,, \tag{A.4.7}$$

while a simple calculation (again simply using the definitions) shows that applying a curl to a curl gives

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}\,. \tag{A.4.8}$$

Vector identities, such as (A.4.5) through (A.4.8), are often easier to prove using notation where a vector's components are listed with indices. That is, since the components of $\nabla \times \mathbf{v}$ are $\epsilon_{ijk}\partial_j v_k$, then the components of $\nabla \times (\nabla \times \mathbf{v})$ has components $\epsilon_{ijk}\epsilon_{klm}\partial_j \partial_l v_m$. This can be simplified using the identities (A.3.7), from which expressions like (A.4.8) are easily derived (compare with the derivation of (A.3.14)).

B Other reading

Here are a few useful upper-level textbooks on classical mechanics.

1. Goldstein, *Classical Mechanics*, Addison-Wesley, 1950.
2. Arnold, *Mathematical Methods of Classical Mechanics*, Springer Graduate Texts in Mathematics, 1989.
3. Landau & Lifshitz, *Mechanics*, Pergamon Press, 1969.
4. Tong, *Classical Dynamics*, Cambridge University Press, 2025 (see also https://www.damtp.cam.ac.uk/user