These notes are meant to accompany the lectures for Physics 1X00 'Special Topics in 1st Year Physics'. The idea of this course is to take several topics given in the 1st-year curriculum and push them a bit further for those with an interest in going a bit deeper into the subjects.

1 Centre of mass, substructure and hierarchies of scale

This section is meant to elaborate on the idea of centre of mass, and the way that its definition captures the *recursiveness* of Newton's Laws. Recursiveness states that under some circumstances we can effectively treat collections of particles as if they were themselves simply point particles.

For instance we can think of everyday-sized objects as macroscopic collections of atoms; we can think of planets, moons and stars as large collections of more everyday-sized objects and we can think of galaxies as large collections of stars. When asking about motion or objects over distances much larger than the size of the objects themselves, we can think of the objects as if they were point particles.

You probably use this intuitive fact all the time, but it reflects a deep property of nature that ensures that we can understand different scales on their own terms. It is one of the basic reasons why science is possible in practice: one does not need to understand everything at once, so we can figure out Newton's Laws without needing to understand atoms or we can figure out atoms without needing to know much about nuclei, and so on.

To outline how this works we start with the usual derivation of the centre of mass for a system of N particles.

1.1 Centre of mass

Suppose the position vector of each particle (as a function of time, t) is denoted by $\mathbf{r}_i(t)$, where $i = 1, ..., N$ labels which particle is which. We denote time derivatives with over-dots, so the velocity of each particle is $\mathbf{v}_i = \dot{\mathbf{r}}_i$ and its acceleration is $\mathbf{a}_i = \dot{\mathbf{v}}_i = \ddot{\mathbf{r}}_i$.

Next divide up the forces acting on particle i into those being applied by one of the other particles. For instance if the force is applied on particle i by particle j we call it $\mathbf{F}_{ij}$. We know these forces satisfy Newton's 3rd Law:

$$\mathbf{F}_{ij} = \mathbf{F}_{ji}\,. \tag{1.1}$$

We also could have other forces from 'outside', such as the gravitational force attracting all N particles towards the Earth. Denote the total force of this type acting on particle i by $\mathbf{F}_i$.

With these labels, the expression of Newton's 2nd Law for each of the N particles amounts to the following N vector equations:

$$\begin{aligned} m_1\,\ddot{\mathbf{r}}_1 &= \mathbf{F}_{12} + \mathbf{F}_{13} + \cdots + \mathbf{F}_{1N-1} + \mathbf{F}_{1N} \\ m_2\,\ddot{\mathbf{r}}_2 &= \mathbf{F}_{21} + \mathbf{F}_{23} + \cdots + \mathbf{F}_{2N-1} + \mathbf{F}_{2N} \\ m_3\,\ddot{\mathbf{r}}_3 &= \mathbf{F}_{31} + \mathbf{F}_{32} + \cdots + \mathbf{F}_{3N-1} + \mathbf{F}_{3N} \\ \vdots &= \vdots \quad \vdots \quad \vdots \quad \vdots \\ m_N\,\ddot{\mathbf{r}}_N &= \mathbf{F}_{N1} + \mathbf{F}_{N2} + \mathbf{F}_{N3} \cdots + \mathbf{F}_{NN-1}\,, \end{aligned} \tag{1.2}$$

where m_i denotes the inertial mass of the particle with label i. These can also be more compactly written collectively, as

$$m_i\,\ddot{\mathbf{r}}_i = \mathbf{F}_i + \sum_{j\neq i} \mathbf{F}_{ij}\,, \tag{1.3}$$

for each i. Once the $\mathbf{F}_i$ and $\mathbf{F}_{ij}$ are expressed as explicit functions of the $\mathbf{r}_i$'s (such as if they arise from Coulomb's Law or Newton's Law or some other explicit form) then (1.2) becomes a collection of N coupled ordinary differential equations for the N unknown functions $\mathbf{r}_i(t)$. Mathematically speaking the problem is solved once these equations are integrated.

The logic leading to the definition of centre of mass starts by adding up these N equations to get their vector sum:

$$\sum_i m_i\,\ddot{\mathbf{r}}_i = \sum_i \mathbf{F}_i + \sum_{\substack{ij\\ \text{with}\, j\neq i}} \mathbf{F}_{ij} = \sum_i \mathbf{F}_i\,, \tag{1.4}$$

where the second equality uses Newton's 3rd Law, (1.1), to cancel everywhere $\mathbf{F}_{ij}$ against $\mathbf{F}_{ji}$. (That they all cancel is perhaps more easily seen when adding up the rows of (1.2).

Wouldn't it be nice if there were some choice of $\mathbf{R}$ for which the left-hand side of this last equation had the form $M\,\ddot{\mathbf{R}}$ for

$$M = \sum_i m_i \tag{1.5}$$

and some choice for $\mathbf{R}$? If so it would also look like Newton's 2nd Law written for the 'position' vector $\mathbf{R}$. Of course there *is* such a choice for $\mathbf{R}$ and it is the usual definition of the centre of mass:

$$\mathbf{R} := \frac{1}{M} \sum_i m_i\,\mathbf{r}_i\,, \tag{1.6}$$

provided all of the masses are time-independent (as we henceforth assume), since direct differentiation gives

$$\dot{\mathbf{R}} = \frac{1}{M} \sum_i m_i\,\dot{\mathbf{r}}_i \qquad \text{and} \qquad \ddot{\mathbf{R}} = \frac{1}{M} \sum_i m_i\,\ddot{\mathbf{r}}_i\,. \tag{1.7}$$

Using this then allows (1.4) to be written

$$M\ddot{\mathbf{R}} = \sum_i \mathbf{F}_i\,, \tag{1.8}$$

showing that the centre of mass responds as if all N particles were grouped together into a point object with mass M, acted on only by the total 'external' force: $\mathbf{F}_{\rm tot} = \sum_i \mathbf{F}_i$. So far so good.

1.2 Substructure and recursiveness

The above arguments have an important feature: they are *recursive*. That is, we can imagine dividing up the N objects into I collections of smaller groupings of particles, which we label using letters from the beginning of the alphabet: $a, b, c = 1, ..., n$. So particles with labels $i = 1, 2, 3, ..., n_a$ are grouped into the collection $a = 1$, those with labels $i = n_a + 1, ..., n_a + n_b$ lie in the group $a = 2$, and so on through to the group with label $a = I$. The grouping labelled by a contains in this way n_a particles, so that

$$\sum_{a=1}^{I} n_a = N \,. \tag{1.9}$$

The total mass within each of these groupings is similarly given by

$$M_a = \sum_{i \in a} m_i \,. \tag{1.10}$$

You can imagine the original sum over i, j was over all of the atoms in an object, say a chair. The groupings might then be different parts of the chair, such as its legs, its seat and its back. But the point is that the argument about to be made works regardless of how you choose to group things, so long as you are always dividing the initial particles up into disjoint sets whose union includes all N particles.

The point of recursiveness is that we can repeat the above process of summing Newton's 2nd Law, but this time only sum over the particles within any particular group. In particular, if we define the centre of mass for grouping a in the way suggested by the above story

$$\mathbf{R}_a := \frac{1}{M_a} \sum_{i \in a} m_i \mathbf{r}_i \,, \tag{1.11}$$

then the result for summing (1.4) over the particles in grouping a is

$$M_a \, \ddot{\mathbf{R}}_a = \mathbf{F}_a + \sum_{b \neq a} \mathbf{F}_b \,, \tag{1.12}$$

where the total external force acting on grouping a is

$$\mathbf{F}_a := \sum_{i \in a} \mathbf{F}_i \,, \tag{1.13}$$

and the total force acting on the grouping a due to the particles in grouping b is

$$\mathbf{F}_{ab} := \sum_{i \in a} \sum_{j \in b} \mathbf{F}_{ij} \,, \tag{1.14}$$

so that

$$\sum_{b \neq a} \mathbf{F}_{ab} = \sum_{i \in a} \sum_{j \notin a} \mathbf{F}_{ij} = \sum_{i \in a} \sum_{j \neq i} \mathbf{F}_{ij} \,. \tag{1.15}$$

The last equality here relies on Newton's 3rd Law, which ensures all pairs of forces for particles completely within grouping a cancel out: $\sum_{i \in a} \sum_{j \neq i \in a} \mathbf{F}_{ij} = 0$.

The point of the exercise is that (1.14) has precisely the same form as did (1.4), with the only difference being that the sum is now only over the groupings of particles and not over the individual particles themselves. Not surprisingly, this means that the total centre of mass can be written in terms of those for each of the groupings in a formula similar to (1.6):

$$\mathbf{R} = \frac{1}{M} \sum_a M_a \, \mathbf{R}_a \,, \tag{1.16}$$

and the equivalence of this to (1.6) once (1.11) is used can be seen by writing

$$M\mathbf{R} = \sum_a M_a \, \mathbf{R}_a = \sum_a \sum_{i \in a} m_i \, \mathbf{r}_i = \sum_i m_i \mathbf{r}_i \,. \tag{1.17}$$

So the upshot is that you did not need to start with particles at all. You could use any convenient grouping of any useful subcomponents of an object and Newton's 2nd Law in the form (1.12) always holds, provided only that the 'position' of a grouping is meant more precisely to be its center of mass. Newton's Laws in themselves cannot help identify what is a constituent and what is built from smaller things (provided one cannot distinguish positions within the grouping well enough to distinguish them from its centre of mass).

1.3 Hierarchies of scale

The above arguments suggest that small but complicated things should tend to be indistinguishable from point particles (and so look 'elementary') provided one never resolves their size. But there really is a missing step in this conclusion, because to complete the program we normally also must express the forces on the right-hand side in terms of the positions, and knowing things like $\mathbf{F}_{ij}$ and $\mathbf{F}_i$ as functions of the $\mathbf{r}_i$ does not immediately give us things like $\mathbf{F}_{ab}$ and $\mathbf{F}_a$ as functions of the $\mathbf{R}_a$. Without knowing this we cannot regard (1.12) as a set of self-contained differential equations for the unknowns $\mathbf{R}_a$, and so cannot solve them to determine $\mathbf{R}_a(t)$. How is this missing step filled in?

Now we come to how to express the forces involved in terms of the positions of the groupings, $\mathbf{R}_a$, rather than the individual particles, $\mathbf{r}_i$. The idea here is that the difference between the $\mathbf{r}_i$'s and $\mathbf{R}_a$ can be negligible if we are interested in forces acting over distances much larger than the size of the groupings themselves. Consider first the simplest example: a constant force acting on our collection of particles, such as the force of gravity at the Earth's surface. In this case

$$\begin{aligned} \mathbf{F} &= m_1 \mathbf{g} + m_2 \mathbf{g} + \cdots + m_N \mathbf{g} = \mathbf{g} \sum_i m_i \\ &= \mathbf{g} \sum_a \sum_{i \in a} m_i = \mathbf{g} \sum_a M_a = M\mathbf{g} \,. \end{aligned} \tag{1.18}$$

Such a constant force can be equally thought of as acting on each particle, i, or acting on each grouping, a, or on the total system, each time proportional to the corresponding mass.

A similar story goes through, say, for the force due to a constant electric field, $\mathbf{E}$, (if we suppose the particles carry an electric charge, q_i). Then

$$\begin{aligned} \mathbf{F} &= q_1\mathbf{E} + \cdots + q_{\scriptscriptstyle N}\mathbf{E} = \sum_i q_i\mathbf{E} \\ &= \sum_a Q_a\mathbf{E} = Q\mathbf{E} \\ \text{where} \quad Q_a &= \sum_{i\in a} q_i \qquad \text{and} \qquad Q = \sum_i q_i = \sum_a Q_a \,. \end{aligned} \tag{1.19}$$

More interesting is when the applied force differs in magnitude and direction at the position of the different particles. For concreteness consider again charged particles experiencing an external Coulomb force due to an external point charge, located at $\mathbf{r} = 0$ with charge $\mathcal{Q}$. For each particle the Coulomb force applied by this external charge is

$$\mathbf{F}_i = q_i\mathcal{Q}\,\frac{\hat{\mathbf{r}}_i}{r_i^2} = q_i\mathcal{Q}\,\frac{\mathbf{r}_i}{r_i^3}\,, \tag{1.20}$$

where $\hat{\mathbf{r}}_i = \mathbf{r}_i/r_i$ (with $r_i = |\mathbf{r}_i|$) is the unit vector pointing towards particle i from the position (*i.e.* $\mathbf{r} = 0$) of the external charge.

Suppose next that the particles are much closer to each other (and so also much closer than their centre of mass) than they are to the external charge. That is, if $\mathbf{R} = M^{-1}\sum_i m_i\mathbf{r}_i$ and $\mathbf{r}_i = \mathbf{R} + \mathbf{x}_i$, then suppose $R = |\mathbf{R}| \gg |\mathbf{r}_i - \mathbf{R}| = |\mathbf{x}_i| = x_i$, for all i. In this case it is possible to Taylor expand $\mathbf{F}_i$ in powers of the small quantity, x_i/R. Using

$$r_i^2 = \mathbf{r}_i\cdot\mathbf{r}_i = (\mathbf{R}+\mathbf{x}_i)\cdot(\mathbf{R}+\mathbf{x}_i) = R^2 + 2\,\mathbf{R}\cdot\mathbf{x}_i + x_i^2 = R^2\left(1 + \frac{2\,\mathbf{R}\cdot\mathbf{x}_i}{R^2} + \frac{x_i^2}{R^2}\right)\,, \tag{1.21}$$

together with the usual Taylor-expansion formula, valid for sufficiently small $|y|$:

$$f(x+y) = \sum_{n=0}^{\infty}\frac{y^n}{n!}\,\frac{\mathrm{d}^n f}{\mathrm{d}x^n}(x) = f(x) + f'(x)y + \frac{1}{2}\,f''(x)\,y^2 + \cdots\,, \tag{1.22}$$

where f' denotes $\mathrm{d}f/\mathrm{d}x$ and $f'' = \mathrm{d}^2 f/\mathrm{d}x^2$ and so on.

Specializing to $f(x+y) = (x+y)^p$ and $x = 1$, this in particular implies $(1+y)^p = 1 + p\,y + \frac{1}{2}\,p(p-1)\,y^2 + \cdots$, and so applying this with $p = -3/2$ and $y = (2\mathbf{R}\cdot\mathbf{x}_i)/R^2 + (x_i^2/R^2)$then gives

$$\begin{aligned} \frac{1}{r_i^3} &= \frac{1}{R^3}\left(1 + \frac{2\,\mathbf{R}\cdot\mathbf{x}_i}{R^2} + \frac{x_i^2}{R^2}\right)^{-3/2} \\ &= \frac{1}{R^3}\left[1 - \frac{3\,\mathbf{R}\cdot\mathbf{x}_i}{R^2} - \frac{3x_i^2}{2R^2} + \frac{15}{8}\left(\frac{2\,\mathbf{R}\cdot\mathbf{x}_i}{R^2}\right)^2 + \mathcal{O}\left(\frac{x_i^3}{R^3}\right)\right] \\ &= \frac{1}{R^3}\left[1 - \frac{3x_i}{R}(\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_i) - \frac{3x_i^2}{2R^2}\left[1 - 5(\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_i)^2\right] + \mathcal{O}\left(\frac{x_i^3}{R^3}\right)\right]\,. \end{aligned} \tag{1.23}$$

Here $\hat{\mathbf{R}} = \mathbf{R}/R$ and $\hat{\mathbf{x}}_i = \mathbf{x}_i/x_i$ are unit vectors respectively pointing in the direction of $\mathbf{R}$ and $\mathbf{x}_i$, and so $\hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_i = \cos\theta_i$ where θ_i is the angle between $\mathbf{R}$ and $\mathbf{x}_i$. The final symbol $\mathcal{O}(\epsilon)$ is a catchall for any terms that are of order ϵ or smaller.

Using this in (1.20) then allows the applied force to be written

$$\begin{aligned} \mathbf{F}_i &= \frac{q_i \mathcal{Q}}{r_i^3}\,\mathbf{r}_i = \frac{q_i \mathcal{Q}}{r_i^3}\Big(\mathbf{R} + \mathbf{x}_i\Big) \\ &= \frac{q_i \mathcal{Q}}{R^2}\left(\hat{\mathbf{R}} + \frac{\mathbf{x}_i}{R}\right)\left[1 - \frac{3x_i}{R}(\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_i) - \frac{3x_i^2}{2R^2}\Big[1 - 5(\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_i)^2\Big] + \mathcal{O}\left(\frac{x_i^3}{R^3}\right)\right] \\ &= \frac{q_i \mathcal{Q}}{R^2}\left\{\hat{\mathbf{R}} + \frac{x_i}{R}\Big[\hat{\mathbf{x}}_i - 3\hat{\mathbf{R}}(\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_i)\Big] + \mathcal{O}\left(\frac{x_i^2}{R^2}\right)\right\}, \end{aligned} \tag{1.24}$$

and so on. To get the total external force applied to grouping 'a' we sum this over all the particles inside this grouping, and use $\mathbf{R} = \mathbf{R}_a$:

$$\mathbf{F}_a = \sum_{i\in a}\mathbf{F}_i = \frac{Q_a\mathcal{Q}}{R_a^2}\,\hat{\mathbf{R}}_a + \frac{\mathcal{Q}}{R_a^3}\Big[\mathbf{D}_a - 3\hat{\mathbf{R}}_a(\hat{\mathbf{R}}_a\cdot\mathbf{D}_a)\Big] + \mathcal{O}\left(\frac{\mathcal{Q}x_i^2}{R_a^4}\right), \tag{1.25}$$

where the *electric dipole moment* of collection 'a' is defined by

$$\mathbf{D}_a := \sum_{i\in a} q_i\,\mathbf{x}_i\,. \tag{1.26}$$

A similar argument goes through for the forces $\mathbf{F}_{ab}$, which for Coulomb forces can be written as

$$\mathbf{F}_{ab} = \frac{Q_aQ_b}{R_{ab}^2}\,\hat{\mathbf{R}}_{ab} + (\text{higher orders in } x_i/R_{ab})\,, \tag{1.27}$$

where $\mathbf{R}_{ab} := \mathbf{R}_a - \mathbf{R}_b$.

These last formulae solve the problem of expressing $\mathbf{F}_a$ as a function of $\mathbf{R}_a$, as needed to allow Newton's 2nd Law to be integrated (in principle) to determine $\mathbf{R}_a(t)$. They show that if we can neglect the size of the groupings relative to the distances between them then the forces between them are precisely the same (*i.e.* the Coulomb force for a point charge) as they were for the particles, but with $q_i \to Q_a$ and $\mathbf{r}_i \to \mathbf{R}_a$.

They also show that if we can include effects accurately enough to see that x_i/R_a is nonzero, this can be done quite efficiently as a series in x_i/R_a and keeping only the first few terms should suffice so long as $x_i \ll R_a$ for all $i \in a$. Including more and more terms in this series requires knowing more and more about the precise distribution of charges inside, and the quantity appearing as the coefficient in the term proportional to x_i^n/R^{2+n} is called the nth *multipole moment*. For $n = 0$ this is the 'monopole moment' or total charge, $Q_a = \sum_{i\in a} q_i$; for $n = 1$ this is the dipole moment, $\mathbf{D}_a = \sum_{i\in a} q_i\mathbf{x}_i$; the term for $n = 2$ is called the 'quadrupole moment' and involves a sum over q_i and two powers of $\mathbf{x}_i$; and so on.

Energy and conservative forces

Although we can make the argument directly in terms of the forces, it is a bit simpler if we restrict our attention only to conservative forces (*i.e.* those forces that arise as the gradient of a scalar potential). For the external forces this means $\mathbf{F}_i = -\nabla_i V$ where $V = V(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N)$ is the external potential and $\nabla_i V$ is the vector whose three components are $\partial V/\partial x_i$, $\partial V/\partial y_i$ and $\partial V/\partial z_i$. For the inter-particle forces this means $\mathbf{F}_i = -\nabla_i U$ where $U = U(r_{ij})$ for $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$. Notice that having $U = U(r_{ij})$ automatically means (because of the chain rule) $\nabla_i U = -\nabla_j U$ and so ensures Newton's 3rd Law is true: $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$.

Aside:

Remember that a force of the form $\mathbf{F} = -\nabla U$ is called conservative because it allows the derivation of a conserved energy. To see this we take the dot product between each row of (1.2) with the corresponding velocity vector, and then add up all the rows. Or equivalently take the dot product between $\dot{\mathbf{r}}_i$ and (1.4) and sum over i. Then we find:

$$\begin{aligned}
\sum_i m_i\, \dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i &= \sum_i \left[\dot{\mathbf{r}}_i \cdot \mathbf{F}_i + \sum_{j\neq i} \dot{\mathbf{r}}_i \cdot \mathbf{F}_{ij} \right] \\
&= \sum_i \dot{\mathbf{r}}_i \cdot \mathbf{F}_i + \sum_{\substack{ij \\ \text{with } j\neq i}} \left[\dot{\mathbf{r}}_i \cdot \mathbf{F}_{ij} + \dot{\mathbf{r}}_j \cdot \mathbf{F}_{ji} \right] \\
&= -\sum_i \dot{\mathbf{r}}_i \cdot \nabla_i V - \sum_{\substack{ij \\ \text{with } j>i}} \left[\dot{\mathbf{r}}_i \cdot \nabla_i U(r_{ij}) + \dot{\mathbf{r}}_j \cdot \nabla_j U(r_{ij}) \right] \\
&= -\dot{V} - \sum_{\substack{ij \\ \text{with } j>i}} \dot{U}(r_{ij})\,, \qquad (1.28)
\end{aligned}$$

which uses the chain rule, that states that the time derivative of any function, $V(\mathbf{r}_1, \cdots, \mathbf{r}_N)$, of the $\mathbf{r}_i(t)$ is given by

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\mathrm{d}x_1}{\mathrm{d}t}\frac{\partial V}{\partial x_1} + \frac{\mathrm{d}y_1}{\mathrm{d}t}\frac{\partial V}{\partial y_1} + \frac{\mathrm{d}z_1}{\mathrm{d}t}\frac{\partial V}{\partial z_1} + \cdots + \frac{\mathrm{d}x_N}{\mathrm{d}t}\frac{\partial V}{\partial x_N} + \frac{\mathrm{d}y_N}{\mathrm{d}t}\frac{\partial V}{\partial y_N} + \frac{\mathrm{d}z_N}{\mathrm{d}t}\frac{\partial V}{\partial z_N} \\
&= \dot{\mathbf{r}}_1 \cdot \nabla_1 V + \cdots + \mathbf{r}_N \cdot \nabla_N V = \sum_i \dot{\mathbf{r}}_i \cdot \nabla_i V\,, \qquad (1.29)
\end{aligned}$$

and similarly, for $U(\mathbf{r}_i - \mathbf{r}_j)$,

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \dot{\mathbf{r}}_i \cdot \nabla_i U + \dot{\mathbf{r}}_j \cdot \nabla_j U\,. \qquad (1.30)$$

The last line of (1.28) can be written as a conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_i \frac{1}{2} m_i\, \dot{\mathbf{r}}_i \cdot \mathbf{r}_i + V + \sum_{j>i} U(r_{ij}) \right] = 0\,, \qquad (1.31)$$

where $\sum_{j>i}$ means sum over both i and j subject to the restriction $j > i$. This last result can be integrated to learn that the quantity inside the square brackets (*i.e.* the energy) is a constant of the motion:

$$\frac{1}{2}\sum_i m_i v_i^2 + V + \sum_{j>i} U(r_{ij}) = E\,. \qquad (1.32)$$

End of aside

Consider again the external force due to a point charge of size $\mathcal{Q}$ situated at the origin. The force due to this charge is conservative since $\mathbf{F}_i = -\nabla_i V$, with V given by the Coulomb potential:

$$V = \sum_i \frac{q_i \mathcal{Q}}{r_i}\,, \tag{1.33}$$

where, as before, $r_i = |\mathbf{r}_i|$. The multipole expansion is much easier to do for V than it is to do for the forces themselves (most things are usually easier to do with the energy, that is why it is useful). Writing $\mathbf{r}_i = \mathbf{R} + \mathbf{x}_i$ as above, we need the Taylor expansion

$$\begin{aligned} \frac{1}{r_i} = \frac{1}{|\mathbf{R}+\mathbf{x}_i|} &= \frac{1}{R}\left(1 + \frac{2\,\mathbf{R}\cdot\mathbf{x}_i}{R^2} + \frac{x_i^2}{R^2}\right)^{-1/2} \\ &= \frac{1}{R}\left[1 - \frac{\mathbf{R}\cdot\mathbf{x}_i}{R^2} - \frac{x_i^2}{2R^2} + \frac{3}{8}\left(\frac{2\,\mathbf{R}\cdot\mathbf{x}_i}{R^2}\right)^2 + \mathcal{O}\left(\frac{x_i^3}{R^3}\right)\right] \\ &= \frac{1}{R}\left[1 - \frac{x_i}{R}(\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_i) - \frac{x_i^2}{2R^2}\left[1 - 3(\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_i)^2\right] + \mathcal{O}\left(\frac{x_i^3}{R^3}\right)\right] ., \end{aligned} \tag{1.34}$$

and so the potential becomes

$$\begin{aligned} V &= \frac{\mathcal{Q}}{R}\sum_i q_i \left[1 - \frac{x_i}{R}(\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_i) - \frac{x_i^2}{2R^2}\left[1 - 3(\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_i)^2\right] + \mathcal{O}\left(\frac{x_i^3}{R^3}\right)\right] \\ &= \frac{Q\mathcal{Q}}{R} - \frac{\mathcal{Q}}{R^2}\,\mathbf{D}\cdot\hat{\mathbf{R}} + \cdots = \frac{Q\mathcal{Q}}{R} - \mathbf{D}\cdot\mathbf{E} + \cdots\,, \end{aligned} \tag{1.35}$$

and the last equality in the last line uses that $\mathbf{E} = \mathcal{Q}\,\hat{\mathbf{R}}/R^2$ is the electric field due to the point charge $\mathcal{Q}$.

Of course there is nothing special about using electrostatic forces in this example, since all that is used in explicit formulae is the inverse-square law. The idea of being able to Taylor expand force in powers of x/R is much broader than this, and would apply to an arbitrary interaction, but our explicit formulae in terms of dipole moments is specific to the inverse-square law. But even these explicit formulae should apply equally well for Newton's inverse-square law of gravity.

But then there is something odd about the above result. For forces between complicated collections of atoms (like the Earth, say) it states that a perfect inverse-square law should apply, but this should only work for interactions between bodies that are much further apart than the size of the bodies themselves. Although that should be fine when thinking about the motions of planets about the Sun, or the Moon about the Earth (see next topic), it should be a terrible approximation for us sitting on the Earth's surface. Yet we shall see that we can calculate the acceleration of gravity at the Earth's surface in terms of the Earth's mass and radius by $g = GM/R^2$, just as we would if the Earth were not right next to us, but was instead a point mass sitting a distance R away from us. How is that possible? The answer to that is part of what the next section is meant to explore.