2 From Newton to Kepler

This topic aims to perform the full derivation of Kepler's Laws (including for elliptical orbits) from Newton's Laws of motion together with Newton's Law of Universal Gravitation. In so doing we completely solve for the motion of two particles that mutually interact through a radially-directed inverse-square law.

This topic is awesome for several reasons. First, it is the first real example a physics student gets to see of a full derivation of the solution to the motion of a system that is not just an artificial problem. So it is the first time you kind of see what a practicing physicist does. But also, it is among the first problems that anybody ever really solved ever! Newton even did his part of inventing calculus partly to work it out.

To top it off, it is the first example of the venerable theme of unification in science. Physics has seen many unifications where two things we thought were very different really turned out to be two sides of the same coin: Maxwell and Faraday brought electricity and magnetism and optics together into the one subject of electromagnetism. Boltzmann and others helped understand Thermodynamics and some of the bulk properties of matter by the properties of atoms that were introduced for completely other reasons in chemistry. The development of quantum mechanics and the understanding of the atom brought much of chemistry within the remit of physics. More recently, in the 1960s and 70s, electromagnetism and the weak interactions were unified into the electro-weak interaction by Glashow, Weinberg and Salam when they built what is now called the Standard Model of particle physics. The possibility that the electro-weak interactions ultimately get unified with the strong and gravitational interactions eventually too has been a strong research directions in modern physics.

But the granddaddy of them all was Newton's unification of the force of gravity, $\mathbf{F} = m\mathbf{g}$, seen to move objects around on the Earth's surface, with the Universal Law of Gravity,

$$\mathbf{F}_{12} = \frac{Gm_1 m_2}{r^2}\,\hat{\mathbf{r}}_{12}\,. \tag{2.1}$$

Here the proposal is that every particle everywhere attracts every other particle everywhere with a force given by (2.1), with $\mathbf{F}_{12}$ being the attractive force acting on body 1 due to body 2 (and, similarly, $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is the force that body 2 experiences due to body 1. $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ is the vector pointing from body 1 to body 2, $r = |\mathbf{r}_{12}|$ is its length and (as usual) hats on a vector denote the unit vector pointing in the same direction: $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$. By making this connection Newton is able to relate the measured value of the gravitational acceleration at the Earth's surface, $g = |\mathbf{g}| = 9.8$ m/s^2 to the mass, M and radius, R, of the Earth, through $g = GM/R^2$ and (because the radius of the Earth had already been inferred by the ancient Greeks) thereby weigh the Earth.

Kepler's Laws

But he doesn't stop there. He also goes on to show that his inverse-square law, (2.1), precisely predicts the 3 Keplerian Laws that had painstakingly been found to summarize planetary orbits through lengthy and detailed observations of the motions of the planets. Kepler's three laws for orbits are

1. Planets move along elliptical orbits, with the Sun located at one of the ellipse's foci;

2. Planets move along these orbits in such a way that the line connecting the Sun to the Planet sweeps out equal areas in equal times. That is, the closer the Planet is to the Sun the faster it moves, and in just such a way as to keep the amount of area swept out a constant.

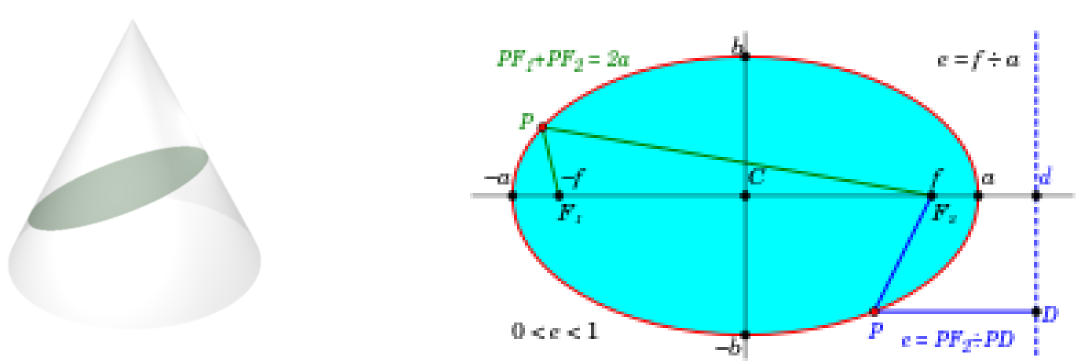
3. One of the two numbers characterizing an ellipse (more about which below) is its semi-major axis, a, defined to be half of the length of the ellipse measured along its longest direction. Kepler's third law states that the *period*, T, of each planetary orbit (*i.e.* the time required to go around the entire orbit exactly once) is related to the orbit's semi-major axis by $a^3/T^2 = k$ where k takes the same value for *any* planet.

By showing that his law for gravity explains Kepler's laws Newton provided overwhelming evidence that his Laws of Motion are not restricted to everyday objects but also apply to the apparent motion of celestial objects in the sky. This essentially gave birth to the field of astrophysics (the physics of celestial objects), and brought astronomy clearly within the scope of physics (in much the same way as the discovery of quantum mechanics would do for chemistry several centuries later). In some ways Newton was to physics as Alexander the Great was to Macedonia: he enormously broadened the domain of what fell under its writ.

Although we now take all this for granted, at the time it was a Big Deal. That is why people to call (2.1) the *Universal* Law of Gravity rather than just the law of gravity.

Refresher course on ellipses

Before showing that Newton proves that Kepler's Laws follow from his law of Universal Gravitation, it is worth boning up a bit on conic sections.



(a) Ellipse as conic section

(b) Ellipse properties

Figure 1: Ellipses can be defined as a kind of conic section (the intersection of a plane with a cone) (left panel), and can be equivalently characterized either by their semi-major axis, a, and semi-minor axis, b (right panel), or by the semi-major axis and eccentricity, e. (Figure source: Wikipedia `https://en.wikipedia.org/wiki/Ellipse`).

Conic sections are curves obtained by the intersection of a plane and a cone and include ellipses, parabolae and hyperbolae (depending on the orientation of the plane relative to the cone). The right-hand panel of Figure 1 provides a more useful way to define an ellipse for our purposes. Take any two points and call them F_1 and F_2. An ellipse can be regarded as the locus of points P for whom the sum of the distances to F_1 and to F_2 is constant whose value is larger than the distance between F_1 and F_2. We call the value of this constant length $2a$, so that all points on the ellipse satisfy $PF_1 + PF_2 = 2a$. The two initial points, F_1 and F_2, are called the *foci* of the ellipse. The distance between the two foci is called $2ea$, where $0 \le e \le 1$ is called the *eccentricity* of the ellipse.

To write the equation for the ellipse choose the origin to lie midway between F_1 and F_2 and choose the x-axis to pass through both foci. Then the defining requirement that any point $P = (x, y)$ lie on the ellipse is

$$PF_1 + PF_2 = 2a = \sqrt{(x-ea)^2 + y^2} + \sqrt{(x+ea)^2 + y^2}\,. \tag{2.2}$$

This equation can be made prettier by moving the first square root over to the other side and squaring both sides of the equation, which after some grouping of terms becomes $a - ex = \sqrt{(x-ea)^2 + y^2}$. Squaring this again then gives $a^2(1-e^2) = x^2(1-e^2) + y^2$, or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,, \tag{2.3}$$

where we define $b = a\sqrt{1-e^2}$. The two numbers a and b are half the length of the ellipse along its longest and shortest directions, respectively (directions called the semi-major and semi-minor axis), as in the figure. a and b (or a and e) completely characterize the properties of any ellipse. In the case $b = a$ (or $e = 0$) the ellipse degenerates into a circle of radius a.

For later purposes it is useful to write the equation of the ellipse in polar coordinates, but with the origin of these coordinates placed at the position of one of the foci. Write, then, $x = z + ea$ so $z = 0$ corresponds to the focus at $x = ea$. Then define polar coordinates by $z = r\cos\theta$ and $y = r\sin\theta$ and plug the result into (2.3). The resulting equation relating r and θ then becomes

$$\frac{(ea + r\cos\theta)^2}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1\,, \tag{2.4}$$

which, after dividing through by r^2, can be rewritten

$$\frac{1}{r^2} = \left(\frac{e}{r} + \frac{\cos\theta}{a}\right)^2 + \frac{\sin^2\theta}{a^2(1-e^2)} = \frac{e^2}{r^2} + \frac{2e\cos\theta}{ar} + \frac{1 - e^2\cos^2\theta}{a^2(1-e^2)}\,. \tag{2.5}$$

Regarding this as a quadratic equation to be solved for $1/r$ then gives the equation for $r(\theta)$

$$\frac{a}{r} = \frac{e\cos\theta + \sqrt{e^2\cos^2\theta + (1 - e^2\cos^2\theta)}}{1-e^2} = \frac{1 + e\cos\theta}{1-e^2}\,, \tag{2.6}$$

where the positive root is chosen to ensure that $r \ge 0$ for all θ. Notice in particular that the points of largest and smallest radii are

$$r_{\min} = r(\theta = 0) = a(1-e) \qquad \text{and} \qquad r_{\max} = r(\theta = \pi) = a(1+e)\,. \tag{2.7}$$

Now we can state Kepler's first two laws with a bit more precision. The first one says that the equation of a planetary orbit is always of the form of (2.6) when written in polar coordinates centred at the Sun's position. What about Kepler's second law? The 2nd law says that equal areas are swept out by the line connecting the Sun to a planet in equal times. But if we compare the position of the planet at a time t with its position at a time $t + \mathrm{d}t$, then it we know its radial distance changes by an amount $\mathrm{d}r$ and its angular position changes by an amount $\mathrm{d}\theta$. We seek the area of the wedge whose radius is r and which subtends the arc-length $\mathrm{d}s = r\mathrm{d}\theta$ as seen by the Sun. This is just $\mathrm{d}A = r \times (r\mathrm{d}\theta) = r^2\mathrm{d}\theta$. So the statement that equal areas get swept out in equal time intervals is equivalent to the statement that

$$\frac{\mathrm{d}A}{\mathrm{d}t} = r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = r^2 \dot{\theta} = \text{constant}\,. \tag{2.8}$$

2.1 Setting up the problem à la Newton

We are interested in solving for the motion of two particles given their mutual interaction through (2.1). Newton's 2nd Law for each particle therefore reads:

$$\begin{aligned} m_1 \ddot{\mathbf{r}}_1 &= +\frac{\kappa}{r^2}\,\hat{\mathbf{r}}_{12} \\ \text{and}\quad m_2 \ddot{\mathbf{r}}_2 &= -\frac{\kappa}{r^2}\,\hat{\mathbf{r}}_{12} = +\frac{\kappa}{r^2}\,\hat{\mathbf{r}}_{21}\,, \end{aligned} \tag{2.9}$$

where we use the short form $\kappa = Gm_1m_2$ and, as usual, dots written over a quantity mean $\mathrm{d}/\mathrm{d}t$ of that quantity and $\hat{\mathbf{r}}_{12}$ is the unit vector pointing from 1 to 2, while $\hat{\mathbf{r}}_{21}$ points from 2 to 1.

The last section told us that the centre-of-mass motion this predicts is simple. That is (assuming $m_1 + m_2 \neq 0$), if $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2)$ then adding the above equations implies $\mathbf{R}$ satisfies

$$\ddot{\mathbf{R}} = 0\,, \tag{2.10}$$

and so the motion of $\mathbf{R}(t)$ over time is easy to predict:

$$\mathbf{R}(t) = \mathbf{R}_0 + \mathbf{V}_0(t - t_0)\,, \tag{2.11}$$

where the initial position and velocity, $\mathbf{R}_0$ and $\mathbf{V}_0$, are the integration constants arising when integrating (2.10) twice.

We wish to now solve the rest of the problem, to determine the relative motion. Since the force depends on the distance between the two bodies it is simplest to use $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{r}_{12}$ as the relative-position coordinate. (This amounts to choosing our origin of coordinates at the position of one of the particles and then tracking from there the position of the other particle.) Once we solve for $\mathbf{r}(t)$ we can invert the definitions of $\mathbf{R}$ and $\mathbf{r}$ to find out how the original variables, $\mathbf{r}_1$ and $\mathbf{r}_2$, evolve. That is

$$\mathbf{r}_1 = \mathbf{R} - \frac{m_2\mathbf{r}}{m_1 + m_2} \qquad \text{and} \qquad \mathbf{r}_2 = \mathbf{R} + \frac{m_1\mathbf{r}}{m_1 + m_2}\,, \tag{2.12}$$

in which we use the solution $\mathbf{R}(t)$ given above, and the solution $\mathbf{r}(t)$ to be found below.

Dividing each of (2.9) by the appropriate mass and subtracting then gives the acceleration equation for $\mathbf{r}$:

$$m\ddot{\mathbf{r}} = -\frac{\kappa}{r^2}\,\hat{\mathbf{r}}\,, \tag{2.13}$$

where m is defined by

$$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2} \qquad \text{or} \qquad m = \frac{m_1 m_2}{m_1 + m_2}\,, \tag{2.14}$$

is called the *reduced mass*. Notice also that the masses only enter into the evolution of $\mathbf{r}$ through the combination

$$\frac{\kappa}{m} = \frac{G m_1 m_2}{m} = G(m_1 + m_2)\,. \tag{2.15}$$

The beauty of (2.13) is that it does not refer to $\mathbf{R}$ and so can be analyzed as if it is a single particle moving under the influence of an externally applied inverse-square-law force. The problem now is to integrate this differential equation twice to learn $\mathbf{r}(t)$ for all times.

2.2 Conservation laws

Although the prospect of solving the nonlinear ordinary differential equation (ODE) given by (2.13) is daunting at first sight, there are sometimes integrations that we can do essentially for free: these are the integrations associated with *conservation laws*.

Conservation of angular momentum

The first trick to help integrate (2.13) is to take its cross product with the position vector $\mathbf{r}$. This leads to the expression

$$\mathbf{r} \times (m\,\ddot{\mathbf{r}}) = -\frac{\kappa}{r^2}\,(\mathbf{r} \times \hat{\mathbf{r}}) = 0\,, \tag{2.16}$$

where the last equality uses that $\mathbf{r}$ is parallel to $\hat{\mathbf{r}}$ and so their cross product vanishes. The good thing here is that the left-hand side of this equation is easy to integrate because it is the derivative of something. To see this notice that using the product rule for differentiating implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathbf{r} \times (m\,\dot{\mathbf{r}})\Big] = \dot{\mathbf{r}} \times (m\,\dot{\mathbf{r}}) + \mathbf{r} \times (m\,\ddot{\mathbf{r}}) = \mathbf{r} \times (m\,\ddot{\mathbf{r}})\,, \tag{2.17}$$

where the first term also vanishes because it is the cross product of a vector, $\dot{\mathbf{r}}$, with a parallel vector, $m\dot{\mathbf{r}}$. So we learn that (2.13) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathbf{r} \times (m\dot{\mathbf{r}})\Big] = 0 \qquad \text{and so} \qquad \mathbf{r} \times (m\,\mathbf{v}) = \mathbf{r} \times \mathbf{p} = \mathbf{L} \quad \text{is } t \text{ independent.} \tag{2.18}$$

Here the three components of $\mathbf{L}$ are the three integration constants found by integrating.

This is great because it is three conserved things, representing 3 pieces of information that do not change with time. Two of these things are easy to interpret. Since $\mathbf{L}$ is a fixed direction in space whose definition ensures it is always perpendicular to both $\mathbf{r}$ and $\mathbf{v} = \dot{\mathbf{r}}$, we see that the motion $\mathbf{r}(t)$ must always be restricted to lie in a plane that is perpendicular to $\mathbf{L}$. We may as well adapt our coordinates to this and choose this plane to be the plane $z = 0$ (*i.e.* the $x-y$ plane), so that $\mathbf{L} = L\,\mathbf{e}_z$ points purely in the z direction with magnitude L. (Here $\mathbf{e}_z$ denotes the unit vector in the z direction, and similarly for $\mathbf{e}_x$ and $\mathbf{e}_y$.)

Once we focus on motion within this plane, the third piece of information is the conserved magnitude of $\mathbf{L}$:

$$\frac{L}{m} = x\,\dot{y} - y\,\dot{x} = \text{constant}\,. \tag{2.19}$$

Conservation of energy

The next easy integral comes because the force whose properties we explore is conservative (*i.e.* is derivable from a potential). To see how this help integrate the equations of motion we next take the dot product of (2.13) with $\mathbf{v} = \dot{\mathbf{r}}$, to get

$$\dot{\mathbf{r}} \cdot (m\,\ddot{\mathbf{r}}) = -\frac{\kappa}{r^2}\,(\dot{\mathbf{r}} \cdot \hat{\mathbf{r}}) = -\frac{\kappa}{r^3}\,(\dot{\mathbf{r}} \cdot \mathbf{r})\,, \tag{2.20}$$

which uses $\hat{\mathbf{r}} = \mathbf{r}/r$.

Again, the good news is that both sides of this equation are the derivatives of something with respect to t. For the left-hand side this is because the product rule for differentiation tells us

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\,\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}\right] = \frac{1}{2}\,(\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}}) = \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}}\,, \tag{2.21}$$

while for the right-hand side it is the same argument, in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\,\mathbf{r} \cdot \mathbf{r}\right] = \frac{1}{2}\,(\dot{\mathbf{r}} \cdot \mathbf{r} + \mathbf{r} \cdot \dot{\mathbf{r}}) = \dot{\mathbf{r}} \cdot \mathbf{r}\,, \tag{2.22}$$

together with

$$-\frac{\kappa}{r^3}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{r^2}{2}\right] = -\kappa\left(\frac{\dot{r}}{r^3}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\kappa}{r}\right)\,. \tag{2.23}$$

Gathering things together, (2.20) is equivalent to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m}{2}\,\mathbf{v} \cdot \mathbf{v}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\kappa}{r}\right)\,, \tag{2.24}$$

which is easy to integrate, and gives the usual expression for energy conservation,

$$\frac{1}{2}\,mv^2 + U(r) = \frac{1}{2}\,mv^2 - \frac{\kappa}{r} = E\,, \tag{2.25}$$

where E is the integration constant and $U(r) := -\kappa/r$ is the inter-particle potential energy.

2.3 Performing the last two integrals

So where do the conservation laws leave us? In eq. (2.13) we had three coupled 2nd-order ordinary differential equations, to be solved for the three components of $\mathbf{r}(t)$. In principle this requires integrating six times (twice for each of the three components of (2.13)). But the conservation laws ($\mathbf{L}$ and E) allowed us to do four of these integrals without even breaking a sweat. Two of them told us we were free to choose coordinates so that the z direction was trivial: $z(t) = 0$ for all t. The remaining motion in the $x - y$ plane involves $x(t)$ and $y(t)$, and these can be found by integrating the two coupled *1st-order* ODEs obtained from the conservation conditions (2.19) and (2.25).

So there are only two more integrals to do in order to completely solve the problem, and the rotational symmetry of the problem suggests that they will be easiest to do in polar coordinates, where we label $\mathbf{r}(t)$ by $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$. Since E and L depend on $\mathbf{v}$ we differentiate:

$$\dot{x} = \dot{r}\,\cos\theta - r\dot{\theta}\,\sin\theta \qquad \text{and} \qquad \dot{y} = \dot{r}\,\sin\theta + r\dot{\theta}\,\cos\theta\,, \tag{2.26}$$

to find

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2\,, \tag{2.27}$$

and

$$L = m(x\dot{y} - y\dot{x}) = mr^2\dot{\theta}\,. \tag{2.28}$$

We see that conservation of angular momentum, as expressed by (2.28), is equivalent to Kepler's 2nd Law — whose precise version is given in (2.8). As such, it does not rely on the precise form of the inverse-square law of gravity, but only on the fact that the potential depends only on r and not on the direction of $\mathbf{r}$.

We are therefore to obtain $r(t)$ and $\theta(t)$ by integrating the following two coupled 1st-order ODEs:

$$\frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{\kappa}{r} = E \tag{2.29}$$

and

$$L = mr^2\dot{\theta}\,, \tag{2.30}$$

with the integration constants L and E regarded as being fixed by the initial conditions. We can decouple $r(t)$ these equations by using (2.30) to eliminate $\dot{\theta}$ in terms of r in eq. (2.29). This leads to a 1st-order equation involving only $r(t)$:

$$\dot{r}^2 + \frac{L^2}{m^2r^2} - \frac{2\kappa}{mr} = \frac{2E}{m}\,, \tag{2.31}$$

or

$$\frac{\mathrm{d}r}{\sqrt{(2E/m) + (2\kappa/mr) - (L/mr)^2}} = \mathrm{d}t\,. \tag{2.32}$$

In principle, this integrates to give

$$\int_{r_0}^{r(t)} \frac{\mathrm{d}r}{\sqrt{(2E/m) + (2\kappa/mr) - (L/mr)^2}} = \int_{t_0}^{t} \mathrm{d}t = (t - t_0)\,. \tag{2.33}$$

Once this has been integrated we use the result for $r(t)$ in (2.30), and then integrate

$$\mathrm{d}\theta = \left(\frac{L}{mr^2}\right)\mathrm{d}t\,, \tag{2.34}$$

to find $\theta(t)$.

In principle we are done (they say the problem is "reduced to quadrature") but in practice we must to the relevant integrals, and although they may always be done numerically they may not be integrable in terms of simple functions in closed form. (We *can* do these integrals we need for the $1/r$ potential, though, as we shall see.)

2.4 The shape of orbits

But more useful for comparing with Keplers Laws would be to know the shape of the orbit, $r(\theta)$, rather than the separate information of $r(t)$ and $\theta(t)$ as functions of t. To do this we use the chain rule of differentiation to relate $\mathrm{d}r/\mathrm{d}\theta$, $\mathrm{d}r/\mathrm{d}t$ and $\mathrm{d}\theta/\mathrm{d}t$ when thinking about $r(t)$ as $r(\theta(t))$. The chain rule in this case states:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\,\frac{\mathrm{d}\theta}{\mathrm{d}t}\,, \tag{2.35}$$

and so

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathrm{d}r/\mathrm{d}t}{\mathrm{d}\theta/\mathrm{d}t} = \frac{\dot{r}}{\dot{\theta}} = \frac{mr^2}{L}\sqrt{(2E/m) + (2\kappa/mr) - (L/mr)^2}\,. \tag{2.36}$$

The orbit therefore satisfies

$$\begin{aligned} \theta - \theta_0 = \int_{\theta_0}^{\theta} \mathrm{d}\theta &= \frac{L}{m}\int_{r_0}^{r} \frac{\mathrm{d}r/r^2}{\sqrt{(2E/m) + (2\kappa/mr) - (L/mr)^2}} \\ &= \int_{u}^{u_0} \frac{\mathrm{d}u}{\sqrt{(2E/m) + 2\kappa u/L - u^2}}\,, \end{aligned} \tag{2.37}$$

where we make the change of variables $u = L/(mr)$.

To do the integral we write it as

$$I = \int \frac{\mathrm{d}u}{\sqrt{(u_+ - u)(u - u_-)}} = \tan^{-1}\left[\frac{u - \frac{1}{2}(u_- + u_+)}{\sqrt{(u_+ - u)(u - u_-)}}\right]\,, \tag{2.38}$$

where comparing powers of u in the square root allows us to read off

$$-u_+u_- = \frac{2E}{m} \qquad \text{and} \qquad u_+ + u_- = \frac{2\kappa}{L}\,. \tag{2.39}$$

For later use these can be solved to give $u_\pm$ as explicit functions of E and L:

$$u_\pm = \frac{\kappa}{L}\left[1 \pm \sqrt{1 + \frac{2EL^2}{\kappa^2 m}}\right]\,. \tag{2.40}$$

In these expressions keep in mind that $u_\pm$ are positive and so they imply E is negative.

When integrating, we are free to choose the origin for θ so that $\theta = 0$ corresponds to $u = u_+$, and so then

$$\begin{aligned} \theta &= \int_0^{\theta} \mathrm{d}\theta = \int_u^{u_+} \frac{\mathrm{d}u}{\sqrt{(u_+ - u)(u - u_-)}} \\ &= \frac{\pi}{2} - \tan^{-1}\left[\frac{u - \frac{1}{2}(u_- + u_+)}{\sqrt{(u_+ - u)(u - u_-)}}\right]\,, \end{aligned} \tag{2.41}$$

and so

$$\frac{u - \frac{1}{2}(u_- + u_+)}{\sqrt{(u_+ - u)(u - u_-)}} = \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta\,, \tag{2.42}$$

Squaring and using $\cot^2\theta + 1 = 1/\sin^2\theta$ then gives (after some grouping of terms)

$$(u_+ - u)(u - u_-) = \frac{1}{4}(u_+ - u_-)^2 \sin^2\theta\,, \tag{2.43}$$

and so solving this quadratic equation for u (and choosing the root so that $u(\theta = 0) = u_+$ gives

$$\frac{1}{r} = \frac{mu}{L} = \frac{m}{2L}\Big[(u_+ + u_-) + (u_+ - u_-)\cos\theta\Big] = \frac{1}{2}\left[\left(\frac{1}{r_+} + \frac{1}{r_-}\right) + \left(\frac{1}{r_+} - \frac{1}{r_-}\right)\cos\theta\right]\,, \tag{2.44}$$

where $r_\pm := L/(mu_\pm)$ and so $u_- \le u \le u_+$ implies $r_{\min} = r_+ \le r \le r_- = r_{\max}$.

This last equation says that $1/r$ is the sum of a constant and something linear in $\cos\theta$, and so comparing with (2.6) shows this is an ellipse. Score another to Newton for getting Kepler's 1st Law. Rewriting (2.6) here for convenience of access

$$\frac{a}{r} = \frac{e\cos\theta + \sqrt{e^2\cos^2\theta + (1 - e^2\cos^2\theta)}}{1 - e^2} = \frac{1 + e\cos\theta}{1 - e^2}\,, \tag{2.45}$$

comparing coefficients allows us to relate (r_+, r_-) or (E, L) to the two parameters of the ellipse (a, e). We find the expected expressions for nearest and furthest approach to the Sun: $r_\pm = (1 \mp e)a$, and this allows us (using (2.39)) to express E and L in terms of the parameters of the ellipse:

$$\frac{L^2}{m^2} = \frac{2\kappa L/m^2}{u_+ + u_-} = \frac{2\kappa/m}{r_+^{-1} + r_-^{-1}} = G(m_1 + m_2)a(1 - e^2) \tag{2.46}$$

and

$$\frac{E}{m} = -\frac{u_+ u_-}{2} = -\frac{L^2}{2m^2}\left(\frac{1}{r_+ r_-}\right) = -\frac{G(m_1 + m_2)}{2a}\,. \tag{2.47}$$

These show how the energy is completely determined by a — with $E = -Gm_1m_2/(2a)$ — while e controls the angular momentum. $e = 1$ corresponds to $L = 0$ and is when the ellipse collapses down to a line. Conversely, the maximum possible angular momentum for a given energy occurs when $e = 0$ and so the orbit is circular.

2.5 Orbital Periods

With Kepler's 1st and 2nd Laws in the bag, all that remains is the 3rd. To this end we need the period, T, of the orbit (defined as the time taken to traverse it exactly once) in terms of the orbital shape, a and e. This is found by integrating (2.33) from $r_{\min}$ to $r_{\max}$ and multiplying by two (since the symmetry of the orbit implies the time taken on each side of the ellipse is the same):

$$\begin{aligned} T &= 2\int_{t_{\min}}^{t_{\max}} \mathrm{d}t = 2\int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{\sqrt{(2E/m) + (2\kappa/mr) - (L/mr)^2}} \\ &= \frac{2L}{m}\int_{u_-}^{u_+} \frac{\mathrm{d}u/u^2}{\sqrt{(u_+ - u)(u - u_-)}} = \left(\frac{2L}{m}\right)\frac{\pi(u_+ + u_-)}{2(u_+u_-)^{3/2}} \\ &= \left(\frac{\pi m}{L}\right)(r_+r_-)^{3/2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right) = \left(\frac{\pi m}{L}\right)\left[a^2(1-e^2)\right]^{3/2}\frac{2}{a(1-e^2)} \\ &= \frac{2\pi\, a^{3/2}}{\sqrt{G(m_1+m_2)}}\,. \end{aligned} \tag{2.48}$$

Equivalently, after squaring this reads

$$\frac{a^3}{T^2} = \frac{G(m_1+m_2)}{4\pi^2}\,. \tag{2.49}$$

This is basically Kepler's 3rd Law since $m_1 + m_2 \simeq M_\odot$ is effectively the same for all planets since they are all so much less massive than the Sun. But not only does Newton recover Kepler's 3rd Law in this way, he adds new information. The constant that they are equal to is related to the mass of the Sun! By measuring a^3/T^2 for the planetary orbits (and measuring G using experiments on Earth) we can solve for $M_\odot$. This is how the Sun (and almost all other astronomical objects) actually get weighed.