2 Faster than Light = Back in Time

This topic is meant to provide an introduction to some of the ideas of Special Relativity, and in particular to argue that the proscription against moving faster than light is equivalent to the forbidding of sending signals to oneself in the past.

2.1 Postulates of Relativity

The starting point for Einstein's reformulation of Newtonian mechanics is a postulate that Newton also uses: that there exist a class of inertial observers for whom particles move in straight lines with constant speed in the absence of any external forces. The rest of Newton's laws (or Einstein's reformulation of them, at least in special relativity) are then aimed at the measurements made by these special observers.

The main two principles from which special relativity follows are given by the following additional assertions.

- **Principle of Relativity:** This principle (also shared by Newton) states that laws of physics as seen by any two inertial observers moving at constant velocity relative to one another are identical. That is, the transformation from one such observer to another is a symmetry of the equations of motion.

- **Universality of the speed of light:** This principle states that every inertial observer measures precisely the same numerical value of the speed of light, $c = 299,792,458$ m/s, regardless of the speed of their motion relative to one another.

It is the second of these that separates Einstein from Newton. It is very counter-intuitive, because our Newtonian intuition says that if observer A measures the speed of a light ray as being c_A, and if observer B moves in the same direction as the light ray with speed (relative to A) of v_{BA}, then observer B should measure the light ray's speed as being c_B with

$$c_A = c_B + v_{BA}\,. \tag{2.1}$$

In particular, the ray should appear to be motionless if observer B moves at the speed of light relative to observer A. (That is, $c_B = 0$ if $v_{BA} = c_A$.)

Furthermore, the addition law, (2.1), seems to be a direct consequence of the addition law for vectors. That is, suppose we measure positions relative to some fixed origin of coordinates at rest relative to some observer A. Further suppose another observer B has position $\mathbf{r}_{BA}$ relative to this origin of coordinates. The position of a particular point on the light wavefront (or of a photon in the light beam) relative to the same origin of coordinates is similarly denoted $\mathbf{r}_{\ell A}$, while the position of the same photon as measured by B is instead called $\mathbf{r}_{\ell B}$. As shown in the figure, these vectors are related by the obvious addition law:

$$\mathbf{r}_{\ell A} = \mathbf{r}_{\ell B} + \mathbf{r}_{BA}\,. \tag{2.2}$$

The Newtonian law for addition of velocities — including (2.1) — then follows by differentiating with respect to t:

$$\mathbf{v}_{\ell A} = \frac{\mathrm{d}\mathbf{r}_{\ell A}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}_{\ell B}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{r}_{BA}}{\mathrm{d}t} = \mathbf{v}_{\ell B} + \mathbf{v}_{BA}\,. \tag{2.3}$$

How can an argument as simple as this be wrong?

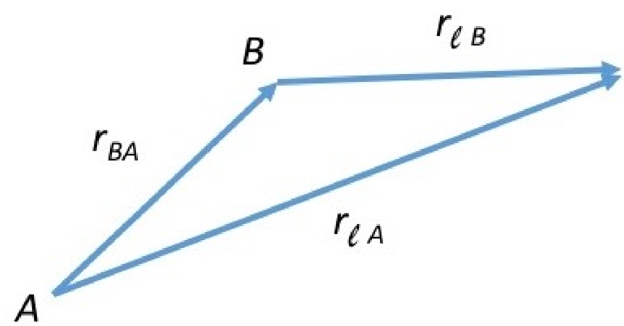


Figure 1: The position of a particle, 'ℓ,' as seen by two observers, A and B, and their relation to the position of B relative to A.

2.2 Lorentz transformations

The fallacy in the above derivation of the addition law for velocities turns to be the hidden assumption that all observers experience the same notion of time. Einstein argued that this must change in precisely the way required to keep the speed of light the same for every observer.

Why would he want to do that? Ultimately he did so because of experiments: people had sought differences in the speed of light as a function of the different velocities experienced by an observer on the Earth's surface due to the Earth's rotation or its orbit around the Sun. In all cases no difference was found, and a famous experiment by Michelson and Morley using an interferometer was particularly convincing on this point.

How would time have to change from observer to observer in order to conspire so that all inertial observers measure precisely the same speed of light? To answer this consider motion in the x direction, and suppose a light ray is measured by observer A to travel a distance $\Delta x = x_2 - x_1$ in a time interval $\Delta t = t_2 - t_1$. The speed measured by A is therefore $c_A = \Delta x/\Delta t$.

How should this look to observer B? The coordinates (x', t') observer B assigns to observer A's coordinates (x, t) must be some linear[1] combination of those measured by A:

$$t' = k_1\, t + k_2\, x \qquad \text{and} \qquad x' = k_3\, t + k_4\, x\,, \tag{2.4}$$

or

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}, \tag{2.5}$$

for some real coefficients k_1, k_2, k_3 and k_4. In Newton's case the coefficients would be

$$k_1 = 1\,, \qquad k_2 = 0\,, \qquad k_3 = v \qquad \text{and} \qquad k_4 = 1\,, \tag{2.6}$$

where v is the relative speed of observers A and B.

[1]The transformation must be linear in order that it preserve vector sums. That is, in our example with one spatial dimension if $x_3 = x_1 + x_2$ then we also want $x_3' = x_1' + x_2'$.

So observer B's version of the measurement of the speed of light would be that the light ray travels a distance $\Delta x'$ in time $\Delta t'$ and so gets $c_B = \Delta x'/\Delta t'$. Given (2.4) and demanding $c_A = c_B = c$ this implies

$$\begin{aligned} 0 &= (\Delta x')^2 - c^2(\Delta t')^2 = (k_3\Delta t + k_4\Delta x)^2 - c^2(k_1\Delta t + k_2\Delta x)^2 \\ &= (k_4^2 - c^2k_2^2)(\Delta x)^2 + (k_3^2 - c^2k_1^2)(\Delta t)^2 + 2(k_3k_4 - c^2k_1k_2)\Delta x\Delta t\,. \end{aligned} \tag{2.7}$$

We determine k_1 through k_4 by demanding the right-hand side be equal to $(\Delta x)^2 - c^2(\Delta t)^2$ and so

$$k_4^2 - c^2k_2^2 = 1\,, \qquad k_3^2 - c^2k_1^2 = -c^2 \qquad \text{and} \qquad k_3k_4 = c^2k_1k_2\,. \tag{2.8}$$

This is a system of 3 equations in 4 unknowns, so we expect a one-parameter family of solutions. The solutions can be written in terms of hyperbolic trig functions:

$$\cosh\beta = \frac{1}{2}(e^\beta + e^{-\beta}) \qquad \text{and} \qquad \sinh\beta = \frac{1}{2}(e^\beta - e^{-\beta})\,, \tag{2.9}$$

which satisfy the identity

$$\cosh^2\beta - \sinh^2\beta = 1\,. \tag{2.10}$$

In terms of this the solutions to (2.8) are

$$k_4 = k_1 = \cosh\beta\,, \qquad \text{and} \qquad \frac{k_3}{c} = c\,k_2 = -\sinh\beta\,, \tag{2.11}$$

where β is a free parameter (that is next related to the relative speed, v, of the two observers).

To relate β to v we use the transformation just identified:

$$\begin{pmatrix} c\,t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} c\,t \\ x \end{pmatrix}\,. \tag{2.12}$$

Specialize this to the trajectory followed by the origin $x' = 0$ of observer B, leading to

$$0 = x' = -c\,t\,\sinh\beta + x\,\cosh\beta\,, \tag{2.13}$$

and since the relative speed of observer B as seen by A is given by $v = x/t$ in the above expression, the relation between β and v is

$$\tanh\beta = \frac{\sinh\beta}{\cosh\beta} = \frac{v}{c}\,. \tag{2.14}$$

Dividing (2.10) by $\cosh^2\beta$ implies

$$\begin{aligned} \cosh^2\beta &= \frac{1}{1-\tanh^2\beta} = \frac{1}{1-v^2/c^2} \\ \text{and so} \qquad \sinh^2\beta &= \cosh^2\beta - 1 = \frac{v^2/c^2}{1-v^2/c^2}\,, \end{aligned} \tag{2.15}$$

and so (2.12) becomes

$$\begin{pmatrix} c\,t' \\ x' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix} \begin{pmatrix} c\,t \\ x \end{pmatrix} \qquad \text{with} \qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}\,. \tag{2.16}$$

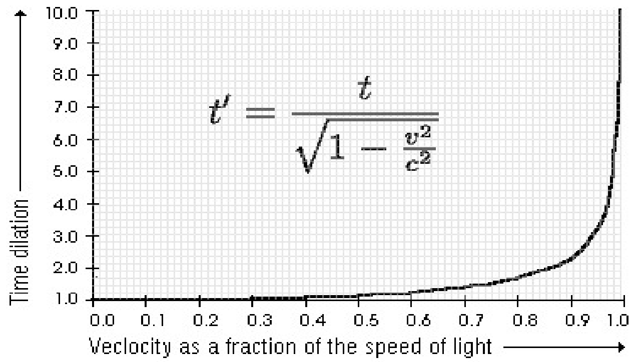


Figure 2: The time-dilation formula relating two observers moving with relative speed v.

The result is the usual Lorentz transformation rule

$$t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}} \qquad \text{and} \qquad x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \,. \tag{2.17}$$

In particular, t measures time along the trajectory of observer A (situated at $x = 0$). The above equations show that this is related to the time, t', measured by observer B by the usual time-dilation formula:

$$t' = \frac{t}{\sqrt{1 - v^2/c^2}} \,, \tag{2.18}$$

as plotted in the figure. Since $t' > t$ moving clocks run more slowly. This is not because of some fault of the clocks. It is because time itself runs more slowly for a moving observer.

Relativistic law for the addition of velocities

We can now derive Einstein's addition law for velocities which replaces the newtonian law (2.1). To do so we perform two successive Lorentz transformations with different speeds, v_1 and v_2 (and so different $\tanh\beta_1 = v_1/c$ and $\tanh\beta_2 = v_2/c$):

$$\begin{pmatrix} c\,t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\beta_1 & -\sinh\beta_1 \\ -\sinh\beta_1 & \cosh\beta_1 \end{pmatrix} \begin{pmatrix} \cosh\beta_2 & -\sinh\beta_2 \\ -\sinh\beta_2 & \cosh\beta_2 \end{pmatrix} \begin{pmatrix} c\,t \\ x \end{pmatrix} = \begin{pmatrix} \cosh\beta_{12} & -\sinh\beta_{12} \\ -\sinh\beta_{12} & \cosh\beta_{12} \end{pmatrix} \begin{pmatrix} c\,t \\ x \end{pmatrix} ,$$

where

$$\begin{aligned} \cosh\beta_{12} &= \cosh\beta_1\cosh\beta_2 + \sinh\beta_1\sinh\beta_2 = \cosh(\beta_1 + \beta_2) \\ \sinh\beta_{12} &= \cosh\beta_1\sinh\beta_2 + \sinh\beta_1\cosh\beta_2 = \sinh(\beta_1 + \beta_2) \,, \end{aligned} \tag{2.19}$$

which uses the addition theorem for hyperbolic trig functions that follow from their definitions. Taking the ratio of these expressions and using (2.14) gives

$$v_{12} = c\tanh\beta_{12} = \frac{c\,(\tanh\beta_2 + \tanh\beta_1)}{1 + \tanh\beta_1\tanh\beta_2} = \frac{v_1 + v_2}{1 + v_1 v_2/c^2} \,. \tag{2.20}$$

This states the relativistic rule for the addition of velocities (in one direction): if B moves relative to A with speed v_1 and C moves in the same direction relative to B with speed v_2 then C moves relative to A with speed v_{12}, given in terms of v_1 and v_2 by (2.20). Notice that if $v_2 = c$ or $v_1 = c$ then $v_{12} = c$, regardless of the other speed. Whenever both v_1 and v_2 are smaller than c then so is v_{12}. Finally notice that Taylor expanding the denominator of (2.20) in powers of $v_1 v_2/c^2$ shows that it reduces to the Newtonian expression (2.1) (as it must) when v_1 and v_2 are both much smaller than c:

$$v_{12} = (v_1 + v_2)\left[1 - \frac{v_1 v_2}{c^2} + \cdots\right] . \tag{2.21}$$

2.3 Relativity of simultaneity and the requirement that $v \le c$

The Lorentz transformations have a very important implication: they imply that different observers disagree on which events in space and time are simultaneous. To see this notice that observer A above regards events with different x but the same value of t to be simultaneous, while observer B regards events with different x' sharing the same t' as being simultaneous. To illustrate this the curves $x = 0$ and $t = 0$ are plotted in the (x, t) plane (labelled Bill), as well as the curves $x' = 0$ and $t' = 0$ (labelled Jim), using the above formulae. With this choice $x' = 0$ corresponds to $x = vt$ while $t' = 0$ represents the curve $t = xv/c^2$, as drawn in Figure 4.

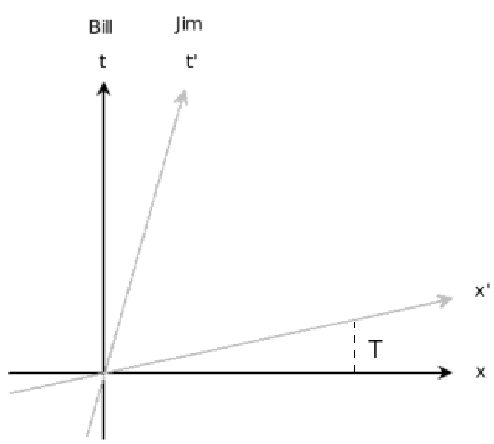


Figure 3: Lines of fixed position and fixed time for two observers.

This kind of figure is called a *space-time diagram*, with time increasing towards the top of the page and spatial lengths extending roughly horizontally. For Bill lines of simultaneous events lie at fixed t and so are parallel to the x axis. For Jim simultaneous events have fixed t' and so are parallel to the x' axis. So there is no one unique observer-independent slicing of space-time up into space and time. This is called 'relativity of simultaneity'. The relativity of simultaneity poses potential problems, since a goal of physics is to predict the future from the past. But how can this be possible if observers cannot agree on what the future, present and past are? A problem of principle blocking the prediction of the future from the past is called a 'causality' problem.

The reason Einstein gets away with it is the proscription in relativity against moving faster than light. So far all we know is that all observers must agree on the value of the speed of light, c, but (so far) nothing prevents things moving faster than c. What gives rise to the requirement that nothing moves faster than light is the causality problem to do with people not all agreeing on what the future and past are. Limiting the speed of travel to luminal and sub-luminal speeds does the job because it turns out that people always agree on the ordering of events that are close enough together that one can get from one to the other while moving only at luminal or sub-luminal speeds.

To see how this works, picture the surface in space-time swept out by a light wave that is emitted in all directions from a particular *event*, P: that is, from a particular place at a given time. This light waves sweeps out a surface in space-time called the light-cone of P. The future light-cone describes all points to which the light wave eventually goes. (Similarly the past light-cone is the set of all points from which an incoming spherical light wave arriving at P would have come.)

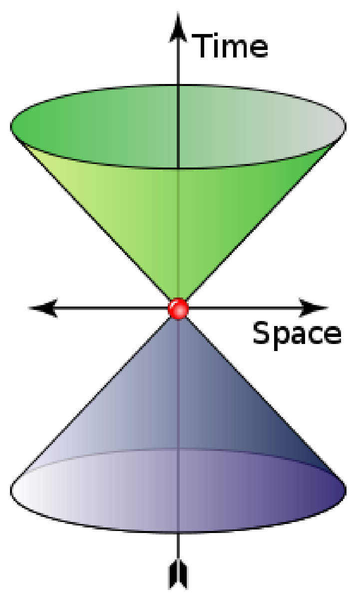


Figure 4: Space-time diagram of the future and past light-cone of an event P.

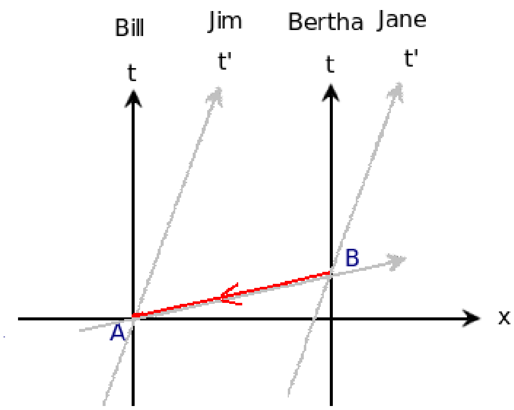
If we choose P as our origin of coordinates, then the light cone is defined by the set of points, $\{t, \mathbf{r}\}$, that satisfy $-c^2t^2 + \mathbf{r}\cdot\mathbf{r} = 0$. All such points are said to be *light-like separated* (or null separated) from P. Similarly, the set of points lying within the interior of the light-cone at P satisfy $-c^2t^2 + \mathbf{r}\cdot\mathbf{r} < 0$ and are said to be *time-like separated* from P. Finally, *space-like separated* points (from P) satisfy $-c^2t^2 + \mathbf{r}\cdot\mathbf{r} > 0$.

What is important is that (as we saw above) the space-time interval (also called the *invariant interval*),

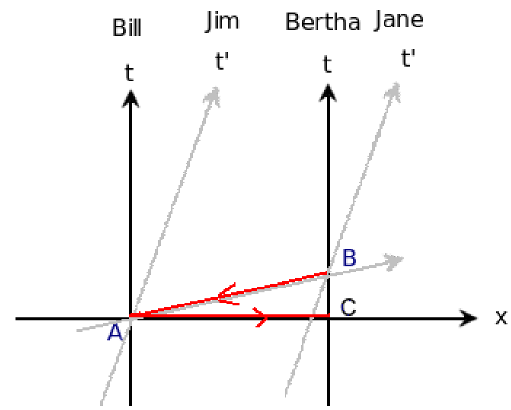
$$(\Delta s)^2 := -c^2(\Delta t)^2 + \Delta\mathbf{r}\cdot\Delta\mathbf{r}\,, \tag{2.22}$$

between two events (where $\Delta t = t_1 - t_2$ and $\Delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$) is *Lorentz-invariant*; that is its value is the same for all inertial observers. (The special case that observers agree on which events form a null interval, $\Delta s = 0$, is the statement that observers agree on the speed of light.) Therefore $(\Delta s)^2 > 0$ for any two events for which there exists an observer for whom they are simultaneous.

Because only events that are space-like separated from P can be regarded as being simultaneous with P for some observers, all observers agree on the temporal ordering of all events separated by null and time-like intervals. But this means that all observers agree on the ordering of all events that can exchange signals travelling at most at the speed of light. The ambiguity of the ordering of space-like separated events does not cause causality problems only because these events can never influence one another, but this only assured if we know that information can travel faster than light.[2]



(a) The first signal from Jane to Jim.



(b) The second signal from Bill to Bertha.

Figure 5: Illustration how two signals sent faster than light between pairs of observers can be arranged to send a signal from one observer to him/herself in their past. Here Jim and Jane are at rest relative to each other but are moving with constant speed relative to Bill and Bertha (who are also at rest relative to one another).

Another way to make the same argument is to show that if signals could be sent faster than light, then it would also be possible for some observers to send signals to themselves backwards in time. The breakdown in causality then manifests itself through the 'Back to the Future' paradox, wherein someone send a message back in time to prevent their own birth taking place.

To see how this happens consider two pairs of inertial observers, where the members of each pair are at rest relative to each other, but where the two pairs move relative to one another with speed v (as in the figure). In the figure Jim and Jane are not moving relative to one another but are moving at constant speed relative to Bill and Bertha. In the left panel

[2]All bets are off once relativity combines with quantum mechanics, however, since then the uncertainty principle means that if you really know you pass exactly through event P, then your momentum is sufficiently uncertain that there is some probability that your speed might be greater than light. The synthesis of relativity and quantum mechanics is nonetheless consistent and causal, but this consistency is very delicate (requiring, for instance, the existence of 'antiparticles' — sharing exactly the same mass and exactly opposite charges — for every existing species of particles).

of the figure Jane sends Jim a signal (from B to A) at precisely the moment that she passes Bertha. By assumption the signal moves much more quickly than light (where infinite speed would connect points that Jim and Jane agree are simultaneous). The signal is received by Jim just as he passes Bill, and in the right panel Bill (who also sees the signal) signals Bertha (from A to C), also at a speed much faster than light (where infinite speed for them would connect points that *they* agree are simultaneous). As the figure shows, the reflected signal arrives back to Bertha (and Jane) before she would have sent the original signal, since C is in the past of B.