1. Consider a decay process where an initial boson decays into a pair of fermions $B \to F_1 + F_2$ (such as occurs in the real world in the examples of the decay of a Higgs boson to an electron and a positron, $H \to e^+e^-$).

   The Schrödinger picture Hamiltonian for this process is: $H = H_{\rm free} + H_{\rm int}$ where

$$H_{\rm free} = \int {\rm d}^3p \Big[\varepsilon_B(p)\, b^\star_{\mathbf p} b_{\mathbf p} + \sum_{\sigma=\pm\frac12} \Big(\varepsilon_F(p)\, a^\star_{\mathbf p\sigma} a_{\mathbf p\sigma} + \varepsilon_F(p)\, \bar a^\star_{\mathbf p\sigma} \bar a_{\mathbf p\sigma}\Big)\Big]\,,$$

   and

$$H_{\rm int} = \sum_{\sigma,\xi=\pm\frac12} \frac{g_{\sigma\xi}}{\sqrt{8(2\pi)^9}} \int \frac{{\rm d}^3p\,{\rm d}^3q\,{\rm d}^3k}{\sqrt{\varepsilon_B(p)}} \Big[b_{\mathbf p}\, a^\star_{\mathbf q\sigma}\, \bar a^\star_{\mathbf k\xi} + \bar a_{\mathbf k\xi}\, a_{\mathbf q\sigma}\, b^\star_{\mathbf p}\Big]\delta^3(\mathbf p - \mathbf q - \mathbf k)\,,$$

   where σ and ξ label the two fermion spin states and $g_{\sigma\xi}$ is a 2-by-2 matrix of couplings. Here $a_{\mathbf p\sigma}$ is the destruction operator for the fermion and $\bar a_{\mathbf p\sigma}$ denotes the destruction operator for its antiparticle. (Bar here does *not* represent hermitian or complex conjugation and you should think of $\bar a$ as simply being the destruction operator for a different particle than the one destroyed by a. Being fermions we have $\{a_{\mathbf p\sigma}, a^\star_{\mathbf q\xi}\} = \{\bar a_{\mathbf p\sigma}, \bar a^\star_{\mathbf q\xi}\} = \delta^3(\mathbf p - \mathbf q)\,\delta_{\sigma\xi}$ and $\{a_{\mathbf p\sigma}, a_{\mathbf q\xi}\} = \{\bar a_{\mathbf p\sigma}, \bar a_{\mathbf q\xi}\} = \{a_{\mathbf p\sigma}, \bar a_{\mathbf q\xi}\} = 0$, while $b_{\mathbf p}$ commutes with all other operators except $[b_{\mathbf p}, b^\star_{\mathbf q}] = \delta^3(\mathbf p - \mathbf q)$. Assume a relativistic dispersion relation

$$\varepsilon_F(p) = \sqrt{\mathbf p^2 + m^2} \quad \text{and} \quad \varepsilon_A(p) = \sqrt{\mathbf p^2 + M^2}\,.$$

   (a) Use Fermi's Golden Rule to compute the differential decay ${\rm d}\Gamma$ as a function of the final fermion momentum for decay in the B rest frame $B \to F(\mathbf q, \sigma) + \overline F(\mathbf k, \xi)$ with specified final fermion spins and momenta lying within ${\rm d}^3q$ and ${\rm d}^3k$ of specified final momenta.

   (b) What is the energy of the outgoing fermion and the energy of the antifermion in the rest frame of the decaying particle as a function of M and m?

   (c) Assume $g_{\sigma\xi} = g_0\,\delta_{\sigma\xi}$. What is the differential decay rate if only the final state fermion and antifermion momenta are measured and their spins are not?

   (d) Perform the integration over final-state momenta to compute the total rate, Γ, for the decay. (For indistinguishable final-state particles is it correct to integrate the final momentum over the usual 4π solid angle?)

   (e) If $M = 125$ GeV and $m = 105$ MeV and $g_0 = (m/v)$ where $v = 246$ GeV, what is the expected mean lifetime, $\tau = 1/\Gamma$, for this decay in seconds? (This choice for g_0 comes because the Higgs boson couples to other particles in proportion to their mass.)

2. Suppose a particle interacts with a classical time-dependent oscillating potential, $V(t) = V_0\cos(\Omega t)$, where V_0 and Ω are positive real numbers, such as is described by the Schrödinger-picture hamiltonian

$$H = E_0 + \int {\rm d}^3p \Big[\omega(p)\, a^\star_p a_p + V(t)(a_p a_{-p} + a^\star_p a^\star_{-p})\Big]\,.$$

   Assume the relativistic dispersion relation $\omega(p) = |\mathbf p|$ and treat the particles as bosons.

(a) Compute the amplitude $\langle k, -k|S|0\rangle$ for the pair-production process in which the no-particle state $|0\rangle$ evolves into a state $|k, -k\rangle = a_k^\star a_{-k}^\star|0\rangle$, treating $V(t)$ in time-dependent perturbation theory to leading nontrivial order in V.

(b) What is the minimum value of Ω for which pair production occurs? Notice that because H is already time-dependent in Schrödinger picture you cannot directly use Fermi's Golden rule since the time-dependence of H implies particle energy is not conserved in this process. (That is, the particle energy is extracted from the energy in the oscillating potential $V(t)$.) So be sure to start your derivation from first principles using time-dependent perturbation theory.