1. Consider a free electrically charged bosons that carry opposite electric charge, $\pm e$ (such as for a π^+ and π^- particle, for example). Suppose the label for single-particle states is chosen to be momentum, and the annihilation operators for the two particles are denoted a_p (for the π^+ say) and $\bar{a}_p$ (for π^-). [Notice that over-bar here does *not* denote complex or hermitian conjugation.] As discussed in the lectures, the Hamiltonian for these particles (when they are not interacting) is

$$H_0 = E_0 + \sum_p \Big[\varepsilon_+(p)\, a_p^\star a_p + \varepsilon_-(p)\, \bar{a}_p^\star \bar{a}_p \Big] \,,$$

where for relativistic particles the single-particle energy must be given by $\varepsilon_\pm(p) = \sqrt{p^2 + m_\pm^2}$ where $m_\pm$ is the rest mass for the $\pi^\pm$ particle. [As we see later in the term, because $\pi^\pm$ are antiparticles for one another, for them a symmetry (CPT) implies $m_+ = m_-$.]

   (a) The electric charge operator for this system is similarly

$$Q_0 = \mathcal{Q}_0 + \sum_p \Big[q_+(p)\, a_p^\star a_p + q_-(p)\, \bar{a}_p^\star \bar{a}_p \Big] \,.$$

   What values must be chosen for the constant $\mathcal{Q}_0$ and the two real positive functions $q_\pm(p)$ if the particle charges are as given above and the no-particle state is electrically neutral? Use your result to calculate the commutator $[H_0\,, Q_0]$.

   (b) Suppose some collection of these particles is prepared in a state $|\chi\rangle$ at $t = 0$. The system's state at a later time t is then given by $|\chi(t)\rangle = U(t)|\chi\rangle$ where $U(t) = \exp[-iH_0 t]$. Use your result for $[H_0\,, Q_0]$ to evaluate $[U(t)\,, Q_0]$. Suppose the initial state is a charge eigenstate: *i.e.* it satisfies $Q_0|\chi\rangle = q_\chi|\chi\rangle$ for some real eigenvalue q_χ. Use your expression for $[U(t)\,, Q_0]$ to evaluate $Q_0|\chi(t)\rangle$ at a later time t. Does it remain a charge eigenstate? If so, what is its charge eigenvalue at time t?

   (c) In real life a single π^+ and a single π^- particle can form an electromagnetic bound state due to their electromagnetic interactions. Suppose the state that is obtained in this way can be written as

$$|\Psi_\ell\rangle = \sum_{pk} \psi_\ell(p,k)\, a_p^\star\, \bar{a}^\star |0\rangle \,,$$

   where $|0\rangle$ is the usual no-particle state and $\psi_\ell(p,k)$ is the amplitude to find each of the particles with momenta p and k given that the bound state has angular momentum quantum number ℓ. Evaluate $\langle\Psi_\ell|H_0|\Psi_\ell\rangle$ and $\langle\Psi_\ell|Q_0|\Psi_\ell\rangle$ in terms of the wave-function $\psi_\ell(p,k)$.

2. Consider a system with two types of free particles but with one particle a boson and the other a fermion. Suppose also that the free hamiltonian is given by[1]

$$H_0 = E_0 + \frac{1}{2}\sum_p \Big[\varepsilon_b(p)\, (a_p^\star a_p + a_p a_p^\star) + \varepsilon_f(p)(c_p^\star c_p - c_p c_p^\star) \Big] \,,$$

[1]This $a^\star a \pm a a^\star$ structure is how hamiltonians will actually arise later in the course.

where a_p destroys the boson and c_p destroys the fermion and p is short-hand for 3-dimensional momentum, $\mathbf{p}$.

(a) In the infinite-volume continuum limit, $\mathcal{V} \to \infty$, it is natural to expect that it is the quantity $E_0/\mathcal{V} = \lambda$ that remains finite. Compute the energy density of the vacuum (*i.e.* the no-particle ground state), $\rho_0 := \langle 0|H_0|0\rangle/\mathcal{V}$, as an integral over $\int \mathrm{d}p$ (but do not yet perform the integral).

(b) Compute the integral if $\varepsilon_b(p) = \sqrt{\mathbf{p}^2 + m_b^2}$ and $\varepsilon_f(p) = \sqrt{p^2 + m_f^2}$ (as appropriate for relativistic particles). Verify that this integral diverges from the limit $p \to \infty$. This divergence can be regularized by limiting the integration range to only $|\mathbf{p}| < \Lambda$ and keeping only the terms in the integral that do not vanish in the limit that $\Lambda \to \infty$. The quantity ρ_0 is a physical thing and so should not depend on artificial things like Λ, but the same is not true of λ_0. How must λ_0 depend on Λ in order to ensure that ρ_0 does not?

(c) What happens to ρ_0 and λ_0 in the special case where $m_b = m_f$? This is the limit that often applies for supersymmetric theories, since for these fermions and bosons have related properties (such as equal masses and couplings).

3. Consider a system of bosons for which the hamiltonian density has the form

$$H = \sum_p \left[\varepsilon(p)\, a_p^\star a_p + \frac{1}{2}\gamma(p)(a_p a_p + a_p^\star a_p^\star) \right] ,$$

where $\varepsilon(p)$ and $\gamma(p)$ are regarded as known real functions and $\varepsilon(p) > 0$. This does not have the usual non-interacting form unless $\gamma(p) = 0$.

(a) Define new creation and annihilation operators, b_p and $b_p^\star$, using

$$a_p := b_p \cosh\beta + b_p^\star \sinh\beta \quad \text{and so} \quad a_p^\star := b_p^\star \cosh\beta + b_p \sinh\beta\,,$$

for some real function $\beta(p)$. What condition must β satisfy to ensure b_p satisfies the same commutation relations, $[b_p, b_q] = 0$ and $[b_p\,, b_q^\star] = \delta_{pq}$, as do the a_p's?

(b) Find a choice for $\beta(p)$ that also ensures H has the free-particle form,

$$H = E_0 + \sum_p E(p)\, b_p^\star b_p\,,$$

when expressed in terms of b_p. What are the values for E_0 and $E(p)$ in terms of the given quantities $\varepsilon(p)$ and $\gamma(p)$? This transformation from a_p to b_p is called a *Bogoliubov* transformation.

(c) Suppose the state $|\Omega\rangle$ is defined by $b_p|\Omega\rangle = 0$ for all p. This state is *not* an eigenstate of $a_p^\star\, a_p$ and so does not have a specific number of the particles counted by $a_p^\star a_p$. Evaluate the mean number of particles $\overline{n}_p = \langle\Omega|a_p^\star\, a_p|\Omega\rangle$ of momentum p in this state.