1. This probem is about how much freedom there is to redefine creation and annihilation operators.

   (a) The usual commutation relation for bosonic annihilation operators is $[a_p, a_q^\star] = \delta_{pq}$ (for discretely normalized states). Suppose someone wanted to define a new destruction operator, b_p, with the definition $b_p := a_p^\star$. What is the commutation relation satisfied by b_q? Suppose there exists a state $|0\rangle$ that satisfies $b_q|0\rangle = 0$. Is this consistent with the commutation relation you found for b_q? (Hint: take its expectation value in the state $|0\rangle$.)

   (b) For fermions the standard commutation relation is instead $\{c_p, c_q^\star\} = \delta_{pq}$. Suppose someone wishes to define a new destruction operator by $\tilde{c}_p := c_p^\star$. Can this choice be consistent with the existence of a state satisfying $\tilde{c}_p|0\rangle = 0$?

2. Consider a system of bosons for which the hamiltonian density has the form

$$H = \sum_p \left[\varepsilon(p)\, a_p^\star a_p + \frac{1}{2}\gamma(p)(a_p a_p + a_p^\star a_p^\star) \right] ,$$

   where $\varepsilon(p)$ and $\gamma(p)$ are regarded as known real functions and $\varepsilon(p) > 0$. This does not have the usual non-interacting form unless $\gamma(p) = 0$.

   (a) Define new creation and annihilation operators, b_p and $b_p^\star$, using

$$a_p := b_p \cosh\beta + b_p^\star \sinh\beta \quad \text{and so} \quad a_p^\star := b_p^\star \cosh\beta + b_p \sinh\beta \,,$$

   for some real function $\beta(p)$. What condition must β satisfy to ensure b_p satisfies the same commutation relations, $[b_p, b_q] = 0$ and $[b_p\,, b_q^\star] = \delta_{pq}$, as do the a_p's?

   (b) Find a choice for $\beta(p)$ that also ensures H has the free-particle form,

$$H = E_0 + \sum_p E(p)\, b_p^\star b_p \,,$$

   when expressed in terms of b_p. What are the values for E_0 and $E(p)$ in terms of the given quantities $\varepsilon(p)$ and $\gamma(p)$? This transformation from a_p to b_p is called a *Bogoliubov* transformation.

   (c) Suppose the state $|\Omega\rangle$ is defined by $b_p|\Omega\rangle = 0$ for all p. This state is *not* an eigenstate of $a_p^\star\, a_p$ and so does not have a specific number of the particles counted by $a_p^\star a_p$. Evaluate the mean number of particles $\overline{n}_p = \langle\Omega|a_p^\star\, a_p|\Omega\rangle$ of momentum p in this state.

3. Consider a harmonic oscillator with Hamiltonian

$$H = \frac{\omega}{2}(a^\star a + a a^\star)$$

   with ω real and a satisfying $[a, a^\star] = 1$ and with energy eigenstates $|n\rangle$ and eigenvalues $E_n = (n + \frac{1}{2})\,\omega$ for $n = 0, 1, 2, \cdots$. The *coherent state* $|\alpha\rangle$ is defined (as in class) by the condition $a|\alpha\rangle = \alpha|\alpha\rangle$. Compute the mean $\langle n|A|n\rangle$ and variance $\langle n|A^2|n\rangle - (\langle n|A|n\rangle)^2$ for the operator $A = \frac{1}{2}(a + a^\star)$ in an energy eigenstate $|n\rangle$. In later sections a quantity like A will play the role of the electromagnetic field, and its fluctuations in the ground state in particular are called vacuum fluctuations.