1. Consider a gas of non-interacting atoms in a cubic atom trap governed by the Hamiltonian

$$H = E_0 + \sum_R \omega_R \, a^\star_R a_R$$

with N collectively labelling single-particle states while a_R satisfies $[a_R, a^\star_M] = \delta_{KM}$ (as usual). The quantities ω_R and $u_R(\mathbf{x})$ are the eigenvalues and orthonormal and complete eigenfunctions that solve

$$\left[-\frac{\nabla^2}{2m} + V(\mathbf{x}) \right] u_R(\mathbf{x}) = \omega_R \, u_R(\mathbf{x}) \, ,$$

with constant potential $V(\mathbf{x}) = V_0$. The trap is defined by demanding the mode functions vanish at the edges of a cube whose sides have length L – *i.e.* by imposing the boundary conditions

$$u_R(\mathbf{x} = 0) = u_R(\mathbf{x} = L\mathbf{e}_x) = u_R(\mathbf{x} = L\mathbf{e}_y) = u_R(\mathbf{x} = L\mathbf{e}_z) = 0 \, ,$$

where $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are the unit vectors pointing in each of the three coordinate directions.

(a) Find the mode functions $u_R(\mathbf{x})$ explicitly, and thereby show that they are labelled by a triple of positive integers: $R = \{r_x, r_y, r_z\}$ where $r_i = 1, 2, 3, \cdots$. What are the eigenvalues $\omega_{r_x r_y r_z}$ given these solutions?

(b) The energy eigenstates are given in the occupation number representation by $|\{N_R\}\rangle$ and have eigenvalues

$$E = E_0 + \sum_R N_R \, \omega_R \, ,$$

for occupation numbers $N_R = 0, 1, 2, \cdots$. What are the four lowest-lying energy eigenstates and eigenvalues if the atoms are assumed to be bosons? What are the four lowest-lying energy eigenstates and eigenvalues if the atoms are fermions?

(c) Define the local field operator as usual by

$$A(\mathbf{x}) = \sum_R a_R \, u_R(\mathbf{x}) \, ,$$

(as in class). Assuming the particles are bosons, what is the mean $\langle A(\mathbf{x}) \rangle$ and correlation function $\langle A^\star(\mathbf{x}) \, A(\mathbf{y}) \rangle - \langle A^\star(\mathbf{x}) \rangle \, \langle A(\mathbf{y}) \rangle$ for the field operator operator $A(\mathbf{x})$ in an energy eigenstate with occupation numbers $\{N_R\}$? (Write your answer as a single sum over the particle label 'R', but do not try to evaluate the mode sum itself in the general case.)

(d) The variance of $\langle A^\star(\mathbf{x}) \, A(\mathbf{x}) \rangle - |\langle A(\mathbf{x}) \rangle|^2$ in the ground state $|0\rangle$ is what is meant by a 'vacuum fluctuation.' Evaluate the mode sum to see what it gives you for the size of these vacuum fluctuations.

(e) Suppose such a system is prepared in a *coherent state* $|\{\alpha_R\}\rangle$, defined (as in class) by the condition $a_P|\{\alpha_R\}\rangle = \alpha_P|\{\alpha_R\}\rangle$. This state is specified once the collection of complex numbers $\{\alpha_R\}$ is specified for each R (much in the same way that an energy eigenstate is specified once a collection of non-negative integers $\{N_R\}$ is specified). What are the mean and variance of $A(\mathbf{x})$ in the coherent state $|\{\alpha_q\}\rangle$? What is the mean and variance of the Hamiltonian H in this state?

(f) Suppose an interaction is turned on so that the Hamiltonian becomes

$$\widetilde{H} = E_0 + J\Big(a_{R_0} + a^\star_{R_0}\Big) + \sum_R \omega_R \, a^\star_R a_R$$

where R_0 is the label of the specific single-particle mode that has the smallest value of ω_R (which we denote by $\omega_{R_0} = \omega_0$). Find explicit expressions for the exact energy eigenstates and eigenvalues for $\widetilde{H}$. (Hint: coherent states will be useful when doing this.)