1. Consider a system involving a species of spinless charged particle interacting electromagnetically with photons. The Hamiltonian for non-interacting charged particles and photons is
$$H_{\rm free} = E_0 + \sum_n \varepsilon_n \, c_n^\star c_n + \sum_{\lambda=\pm} \int {\rm d}^3k \; \omega(k) \, a^\star_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} \,.$$
Here $\lambda = \pm 1$ labels the photon's two helicity states and the photon single-particle energy is $\omega(k) = |\mathbf{k}|$. The charged particles are here imagined to be interacting with a spherically-symmetric harmonic oscillator potential, $V(\mathbf{r}) = \frac{1}{2} M\Omega^2 r^2$, perhaps representing the net force that keeps them in local equilibrium within some sort of atom or molecule. Here M is the charged-particle's mass and Ω is its frequency of oscillation. Because of this their single-particle states are labelled by three non-negative integers, $\{n_x, n_y, n_z\}$, and their energies are given by $\varepsilon_n = (n_x + n_y + n_z + \frac{3}{2})\Omega$. For brevity of notation these three integers are collectively denoted by 'n' in the above sum (and below). Suppose the charged particles are bosons (as are photons) so the creation and annihilation operators obey the standard algebra
$$[c_n \,, c_m^\star] = \delta_{nm} \quad \text{and} \quad [a_{\mathbf{k}\lambda} \,, a^\star_{\mathbf{q}\zeta}] = \delta^3(\mathbf{k} - \mathbf{q})\delta_{\lambda\zeta} \,,$$
together with $[c_n \,, a_{\mathbf{k}\lambda}] = [c_n \,, a^\star_{\mathbf{k}\lambda}] = 0$. The electric charge operator is similarly given by
$$Q = e \sum_n c_n^\star c_n \,.$$

   (a) What is the eigenvalue of energy and of electric charge for the state
   $$|\Psi\rangle \propto (c_n^\star)^{N_n} (c_m^\star)^{N_m} (a^\star_{\mathbf{k}\lambda})^{N_\gamma} |0\rangle \,?$$
   As usual $|0\rangle$ satisfies $a_{\mathbf{k}\lambda}|0\rangle = c_n|0\rangle = 0$.

   (b) Calculate the average energy density, $\langle H_{\rm free}\rangle_T/\mathcal{V}$ and average charge density $\langle Q\rangle_T/\mathcal{V}$ within a thermal ensemble, for which the averages are defined as $\langle \mathcal{O}\rangle_T := {\rm Tr}\,[\rho\,\mathcal{O}]$ for any operator $\mathcal{O}$, and $\rho = Z^{-1}\exp[-H_{\rm free}/T]$ where T is the temperature and Z is chosen to ensure ${\rm Tr}\;\rho = 1$. Here $\mathcal{V}$ is the system volume, and the limit $\mathcal{V} \to \infty$ is imagined so that the result can be performed using continuum-normalized momentum states.

2. Thomson scattering is the low-energy scattering of photons by charged particles, and was used by J.J.Thomson when first investigating the properties of electrons. It can be described by the following local Hamiltonian (that we will later derive from quantum electrodynamics): $H = H_{\rm free} + H_{\rm int}$ where
$$H_{\rm int} = \frac{e^2}{2m} \int {\rm d}^3x \; \mathbf{A} \cdot \mathbf{A} \, \psi^\star \psi \,, \tag{1}$$
where all of the position-space quantum fields are evaluated at position $\mathbf{x}$ (and at time t, in the interaction picture). The charged-particle field, $\psi(\mathbf{x}, t)$, is defined (in the interaction picture) by
$$\psi(\mathbf{x}, t) = \sum_n u_n(\mathbf{x}, t) \; c_n \,,$$

where $u_n(\mathbf{x}, t)$ is the normalized Schrödinger eigenfunction satisfying

$$\left[-\frac{1}{2M}\nabla^2 + \frac{M\Omega^2}{2}(x^2+y^2+z^2)\right] u_n = \varepsilon_n \, u_n \, .$$

The photon field is similarly defined by

$$\mathbf{A}(\mathbf{x}, t) = \sum_{\lambda=\pm} \int \frac{\mathrm{d}^3 k}{\sqrt{(2\pi)^3 2\omega(k)}} \left[\mathbf{e}(\mathbf{k}, \lambda)\, a_{\mathbf{k}\lambda} e^{-i\omega(k)t + i\mathbf{k}\cdot\mathbf{x}} + \mathbf{e}^*(\mathbf{k}, \lambda)\, a^\star_{\mathbf{k}\lambda} e^{i\omega(k)t - i\mathbf{k}\cdot\mathbf{x}}\right] ,$$

where the two polarization vectors $\mathbf{e}(\mathbf{k}, \lambda = \pm)$ are orthonormal [that is they satisfy $\mathbf{e}(\mathbf{k}, \lambda) \cdot \mathbf{e}(\mathbf{k}, \zeta) = \delta_{\lambda\zeta}$] and also satisfy $\mathbf{k} \cdot \mathbf{e}(\mathbf{k}, \lambda) = 0$ for each λ and $\mathbf{k}$. For example, for $\mathbf{k}$ pointed in the z direction one could choose linearly polarized photons for which $\mathbf{e}(\mathbf{k}, \lambda = x) = \mathbf{e}_x$ and $\mathbf{e}(\mathbf{k}, \lambda = y) = \mathbf{e}_y$ to be unit vectors pointing in the x and y directions. Alternatively, circular polarization would correspond to the choices $\mathbf{e}(\mathbf{k}, \lambda = \pm) = \frac{1}{\sqrt{2}}\,(\mathbf{e}_x \pm i\mathbf{e}_y)$.

(a) Prove the following completeness identify satisfied by the components of the photon polarization vectors:

$$\sum_{\lambda=\pm} e_i(\mathbf{k}, \lambda) e^*_j(\mathbf{k}, \lambda) = \delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2} \, .$$

(b) The notes have been writing the interaction Hamiltonian directly in terms of the creation and annihilation operators, with the kind of term responsible for Thompson scattering having the form

$$\hat{H}_{\rm int} = \sum_{\zeta,\xi=\pm} \int \mathrm{d}^3 p \, \mathrm{d}^3 q \, \left[\mathfrak{g}_{mn}(\mathbf{p}, \mathbf{q})\, c^\star_n \, c_m \, a^\star_{\mathbf{p}\zeta} \, a_{\mathbf{q}\xi} + \mathfrak{g}^*_{mn}(\mathbf{p}, \mathbf{q})\, c^\star_m \, c_n \, a^\star_{\mathbf{q}\xi} \, a_{\mathbf{p}\zeta}\right] .$$

Calculate the prediction for $\mathfrak{g}_{mn}(\mathbf{p}, \mathbf{q})$ that is obtained from the local interaction of eq. (1). (You can leave your result expressed as an integration over the mode functions $u_n(\mathbf{x}, t)$ if necessary.)

(c) Use the above result to compute the matrix element

$$\langle C(m), \gamma(\mathbf{q}, \zeta)|H_{\rm int}|C(n), \gamma(\mathbf{k}, \lambda)\rangle \, ,$$

for the scattering process $C(n) + \gamma(\mathbf{k}, \lambda) \to C(m) + \gamma(\mathbf{q}, \zeta)$, assuming all initial and final states are initially already occupied, with N_n and N_m particles occupying the initial and final harmonic oscillator states and with photons described by the phase-space densities $f(\mathbf{k}, \lambda)$ and $f(\mathbf{q}, \zeta)$ (as normalized in the class notes).

(d) Use Fermi's Golden Rule to compute the differential rate $\mathrm{d}\Gamma/\mathrm{d}^3 q$ for elastic scattering as a function of the photon momentum, in the limit that the incident photon energy satisfies $|\mathbf{k}| \ll \sqrt{M\Omega}$. In particular show that the differential rate in this limit is always elastic (*i.e.* vanishes unless $m = n$) and is independent of the functional form of $u_n(\mathbf{x}, t)$ for the initial charged-particle state. [Notice that when $M \gg \Omega$ this limit need *not* also imply $\omega(\mathbf{k}) = |\mathbf{k}| \ll \Omega$, so do not assume this in the above proof.]

(e) When the polarization, λ, of the initial photon is unknown (or experiments are run with a random distribution of initial polarizations) then the squared matrix element that appears in Fermi's Golden Rule is replaced by its average over the initial polarizations:

$$\Big|\langle C(m), \gamma(\mathbf{q}, \zeta)|H_{\rm int}|C(n), \gamma(\mathbf{k}, \lambda)\rangle\Big|^2 \to \frac{1}{2}\sum_{\lambda=\pm}\Big|\langle C(m), \gamma(\mathbf{q}, \zeta)|H_{\rm int}|C(n), \gamma(\mathbf{k}, \lambda)\rangle\Big|^2 .$$

Use this, together with the result from part (a) above, to show that the scattering rate, $\mathrm{d}\Gamma/\mathrm{d}^3q$, for initially unpolarized light is proportional to $\frac{1}{2}[1-\cos^2\beta]$ where β is the angle between the momentum $\mathbf{k}$ of the incoming photon and the polarization $\mathbf{e}(\mathbf{q}, \zeta)$ of the outgoing photon [so $\mathbf{k}\cdot\mathbf{e}(\mathbf{q}, \zeta) = |\mathbf{k}|\cos\beta$].

(f) If the final photon polarization is also not measured, then the total differential rate is obtained by summing the above result over the final photon helicity: $\mathrm{d}\Gamma_{\rm tot} = \sum_{\zeta=\pm}\mathrm{d}\Gamma$. Use this to show that the unpolarized differential cross section, $\mathrm{d}\sigma_{\rm tot}$, for the scattering of a single photon from a single charged particle always has the standard Thomson-scattering form in the limit $|\mathbf{k}| \ll \sqrt{M\Omega}$:

$$\frac{\mathrm{d}\sigma_{\rm tot}}{\mathrm{d}\Omega} = \frac{\alpha^2}{2m^2}\Big(1+\cos^2\theta\Big) ,$$

where $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ is the element of solid angle for the outgoing momentum $\mathbf{q}$ in spherical coordinates (with the incoming momentum $\mathbf{k}$ taken to define the positive z-axis), where $\alpha = e^2/4\pi$ is the fine-structure constant, and θ is the angle between the incoming and outgoing photon directions. What fraction of scattered photons emerge within the equatorial range $45^o < \theta < 135^o$?

(g) Integrate over all angles and show that the total Thompson-scattering cross section is

$$\sigma_{\rm tot} = \frac{8\pi}{3}\left(\frac{\alpha}{m}\right)^2 .$$

If $m \simeq 0.511$ MeV and $\alpha \simeq 1/137$ what is the size of this scattering cross section in barns? (One barn is 10^{-24} cm^2.) For comparison, nuclear scattering cross sections are often of order $2\pi a^2$ with $a \simeq 1$ fm. Is the Thomson cross section larger or smaller than this?