1. Consider a system of bosons that are trapped in a harmonic potential in the $x - y$ plane but free to move in the z direction. The Hamiltonian for the system is

$$H = \int_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z \left[\frac{1}{2m}\nabla\Psi^\star \cdot \nabla\Psi + V(\mathbf{x})\Psi^\star\Psi\right]$$

where

$$V(\mathbf{x}) = \frac{1}{2}m\omega^2(x^2 + y^2)\,,$$

and all products of operators are taken to mean the symmetric combination

$$AB := \frac{1}{2}(AB + BA)\,. \tag{0.1}$$

(a) Show that this Hamiltonian is diagonalized by writing

$$\Psi(x, y, z, t) = \sum_{mn} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\sqrt{2\pi}} \left[\mathfrak{a}_{nmk}\, u_{mn}(x, y)\, e^{-i\omega_{nm}(k)t + ikz}\right]$$

with commutation relations

$$\left[\mathfrak{a}_{mnk}\,, \mathfrak{a}^\star_{rsq}\right] = \delta_{mr}\delta_{ns}\delta(k - q)\,,$$

and mode functions $u_{mn}(x, y)$ satisfying

$$\left[-\frac{1}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + V(\mathbf{x})\right] u_{mnk}(x, y) = \varepsilon_{mn}\, u_{mnk}(x, y)\,,$$

with normalization conditions

$$\int \mathrm{d}x\mathrm{d}y\; u^*_{mn}(x, y)\, u_{rs}(x, y) = \delta_{mr}\delta_{ns}\,.$$

(You do not need to compute the explicit form of these mode functions.)

(b) Show that the single-particle energy eigenvalues are given by

$$\omega_{mn}(k) = \frac{k^2}{2m} + \varepsilon_{mn}$$

with

$$\varepsilon_{mn} = (m + n + 1)\,\omega\,.$$

What are the allowed values for m and n?

(c) Show that the ground state energy per-unit-length in the z direction is

$$\rho_0 = \lim_{L\to\infty} \frac{E_0}{L} = \frac{1}{2}\sum_{mn} \int \frac{\mathrm{d}k}{2\pi}\, \omega_{mn}(k)\,.$$

Evaluate the integral and the mode sums (regulate them using the same trick we did in class) and compute how ρ_0 depends on the parameters m and ω.

You may find the following integral useful:

$$\int_0^\infty \mathrm{d}s \, \frac{s^a \, e^{ws}}{(e^{ws} - 1)^2} = \frac{\Gamma(a+1)}{w^{a+1}} \, \zeta(a) \,,$$

where $\Gamma(z)$ is Euler's factorial function and $\zeta(a)$ is the Riemann ζ-function defined by

$$\zeta(a) := \sum_{n=1}^{\infty} \frac{1}{n^a} \,.$$