1. Consider the hamiltonian

$$H = \int \mathrm{d}^3p \left\{ \varepsilon(\mathbf{p})\, a^\star_{\mathbf{p}}\, a_{\mathbf{p}} + \gamma(\mathbf{p}) \left[a_{\mathbf{p}}\, a_{-\mathbf{p}} + a^\star_{\mathbf{p}}\, a^\star_{-\mathbf{p}} \right] \right\} ,$$

where (as usual) $[a_{\mathbf{p}}\, , a^\star_{\mathbf{q}}] = \delta^3(\mathbf{p} - \mathbf{q})$.

   (a) Perform a Bogoliubov transformation to diagonalize H to write it in the free-particle form

   $$H = E_0 + \int \mathrm{d}^3p\; E(\mathbf{p})\, c^\star_{\mathbf{p}}\, c_{\mathbf{p}} \,.$$

   That is, calculate how $c_{\mathbf{p}}$ is related to $a_{\mathbf{p}}$ and $a^\star_{-\mathbf{p}}$, in order to ensure H has this form and also and verify that $[c_{\mathbf{p}}, c^\star_{\mathbf{q}}] = \delta^3(\mathbf{p} - \mathbf{q})$.

   (b) What are $E(\mathbf{p})$ and E_0 in terms of $\epsilon(\mathbf{p})$ and $\gamma(\mathbf{p})$?

   (c) If $|\Omega\rangle$ is the ground state, defined by $c_{\mathbf{p}}|\Omega\rangle = 0$, what is its energy? What is the mean number of the original particles $\langle\Omega|c^\star_{\mathbf{p}}\, c_{\mathbf{p}}|\Omega\rangle$ in this state?

2. Verify that the electromagnetic energy can be written in the harmonic oscillator form. That is rewrite

$$H = \frac{1}{2} \int \mathrm{d}^3x \left[\mathbf{E}^2 + \mathbf{B}^2 \right]$$

in terms of $a_{\mathbf{p}\lambda}$ using $\mathbf{E} = -\partial_t \mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$ where

$$\mathbf{A}(\mathbf{x}, t) = \sum_{\lambda=\pm1} \int \frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3 2\omega(\mathbf{k})}} \left[a_{\mathbf{k}\lambda} \mathbf{e}_\lambda(\mathbf{k})\, e^{-i\omega(\mathbf{k})t + i\mathbf{k}\cdot\mathbf{x}} + a^\star_{\mathbf{k}\lambda} \mathbf{e}^*_\lambda(\mathbf{k})\, e^{i\omega(\mathbf{k})t - i\mathbf{k}\cdot\mathbf{x}} \right]$$

with $\omega(\mathbf{k}) = |\mathbf{k}|$ and $\mathbf{k}\cdot\mathbf{e}_\lambda(\mathbf{k}) = 0$ and $[a_{\mathbf{k}\lambda}\, , a^\star_{\mathbf{q}\zeta}] = \delta^3(\mathbf{k} - \mathbf{q})\, \delta_{\lambda\zeta}$. Be sure to always use the operator ordering $AB \to \frac{1}{2}\{AB + BA\}$ as discussed in class. Show that the final form of H can be written

$$H = E_0 + \sum_{\lambda=\pm1} \int \mathrm{d}^3k\; |\mathbf{k}|\, a^\star_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} \,.$$

and derive an integral expression for E_0.