1. I claimed in class that Maxwell's equations can be written in manifestly relativistic form if they are expressed in terms of the electromagnetic current 4-vector and the antisymmetric electromagnetic field-strength tensor, defined by

$$\begin{pmatrix} J^0 \\ J^x \\ J^y \\ J^z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{pmatrix} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix},$$

where J^i are the components of the electromagnetic current $\mathbf{J}$, while E_i and B_i are the components of the electric and magnetic fields, and J^0 is the electric charge density. 4-vectors and tensors with indices written as superscripts and subscripts are related using the Minkowski metric, with (for instance) $J^\mu = \eta^{\mu\nu} J_\nu$ and $F^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\beta} F_{\lambda\beta}$ and so on, where $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ and $\eta^{\mu\nu}$ is its inverse matrix (so $\eta^{\mu\nu}\eta_{\nu\lambda} = \delta^\mu_\lambda$). The Einstein summation convention is in force throughout.

   (a) Show that the four Maxwell equations can be written in terms of these as

$$\partial_\mu F^{\mu\nu} + J^\nu = 0 \quad \text{and} \quad \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0\,.$$

   (b) In relativity the energy density is the time-time component of a symmetric tensor $\mathcal{H} = T_{00}$ where for electromagnetism

$$T_{\mu\nu} = F_{\mu\lambda} F_\nu{}^\lambda - \frac{1}{4}\,\eta_{\mu\nu}\, F^{\lambda\rho} F_{\lambda\rho}$$

   Use this to verify that the energy density of an electromagnetic field is $\mathcal{H} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$.

2. Use the Hamiltonian

$$H = \int \mathrm{d}^3x \left[\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + \Phi^\star \left(-\frac{1}{2m}\nabla^2\right)\Phi + \frac{e^2}{2m}(\Phi^\star\Phi)\mathbf{A}\cdot\mathbf{A}\right] \tag{0.1}$$

to compute the leading contribution to the differential Thomson scattering cross section $\mathrm{d}\sigma_\pm/\mathrm{d}\Omega$ for polarized electron-photon scattering,

$$e^-(\mathbf{p}) + \gamma(\mathbf{k}, \lambda) \to e^-(\mathbf{p}') + \gamma(\mathbf{k}', \lambda')$$

as a function of the photon scattering direction (relative to the initial photon direction) and as a function of both photon polarizations. Treat the electron as spinless and imagine that the final electron momentum $\mathbf{p}'$ is not measured. Here $\mathrm{d}\sigma_+$ is the cross section when $\lambda' = +\lambda$ and $\mathrm{d}\sigma_-$ is the cross section where the subscript $\lambda' = -\lambda$. Use the polarization vectors given in class (or the notes).

3. Suppose a field $\phi(x)$ transforms as a Lorentz scalar, and is related to creation and annihiliation operators for its particle and antiparticle by

$$\phi(x) = \int \frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2E_p}}\, \mathfrak{a}_\mathbf{p}\, e^{ip\cdot x} \tag{0.2}$$

where $p\cdot x := \eta_{\mu\nu}p^\mu x^\nu = -E_p\, t + \mathbf{p}\cdot\mathbf{x}$ and the momentum 4-vector has components $p^\mu = \{p^0 = E_p\,, \mathbf{p}\}$ and the position 4-vector is $x^\mu = \{x^0 = t, \mathbf{x}\}$ and $E_p = \sqrt{\mathbf{p}^2 + m^2}$.

(a) Assuming that the particles are bosons, so $[\mathfrak{a}_{\mathbf{p}}, \mathfrak{a}^\star_{\mathbf{q}}] = \delta^3(\mathbf{p} - \mathbf{q})$, evaluate the commutator $[\phi(x)\, \phi^\star(y)]$ as a function of x^μ and y^ν and show that it is a Lorentz invariant function of $(x-y)^2 = \eta_{\mu\nu}(x-y)^\mu(x-y)^\nu$. Does this commutator vanish when $(x-y)^2 > 0$ (so the separation is spacelike)?

(b) Suppose the scalar field $\phi(x)$ is instead supplemented by another scalar field $\psi(x)$, with both fields related to their respective creation and annihiliation operators by

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{\sqrt{(2\pi)^3 2E_p}}\, \mathfrak{a}_{\mathbf{p}}\, e^{ip\cdot x} \quad \text{and} \quad \psi(x) = \int \frac{\mathrm{d}^3 p}{\sqrt{(2\pi)^3 2E_p}}\, \mathfrak{c}_{\mathbf{p}}\, e^{ip\cdot x}\,, \tag{0.3}$$

where $\mathfrak{a}_{\mathbf{p}}$ and $\mathfrak{c}_{\mathbf{p}}$ are the destruction operators for the two types of particles, both of which are bosons. Suppose they both have relativistic dispersion relation: $E_p = \sqrt{\mathbf{p}^2 + m^2}$ but do not assume they have the same masses. Define

$$\Phi(x) = \phi(x) + \psi^\star(x)$$

and evaluate $[\Phi(x)\, \Phi^\star(y)]$ as a function of $(x-y)^2$, assuming as usual $[\mathfrak{a}_{\mathbf{p}}, \mathfrak{a}^\star_{\mathbf{q}}] = \delta^3(\mathbf{p} - \mathbf{q})$ and $[\mathfrak{c}_{\mathbf{p}}, \mathfrak{c}^\star_{\mathbf{q}}] = \delta^3(\mathbf{p} - \mathbf{q})$ (with all other commutators, like $[\mathfrak{a}_{\mathbf{p}}, \mathfrak{c}_{\mathbf{q}}] = 0$ and so on, vanishing). Is there a choice for the two particle masses for which the commutator $[\Phi(x)\, \Phi^\star(y)]$ vanishes for all spacelike separations?

(c) Repeat the previous calculation but this time assuming that the two types of particles are fermions rather than bosons. Is it possible in this case to choose the masses for the particles so that either the commutator $[\Phi(x), \Phi^\star(y)]$ or the anticommutator $\{\Phi(x), \Phi^\star(y)\}$ vanishes when x and y are spacelike separated?