1. Consider an electromagnetic field in a waveguide defined as a rectangular region $0 \le x \le a$ and $0 \le y \le b$ and $-\infty < z < \infty$, whose cross-sectional area is $\mathcal{A} = ab$. Suppose the electromagnetic field must satisfy the boundary conditions $\mathbf{A}(x, y, z, t) = 0$ on the boundaries defined by $x = 0$, $x = a$, $y = 0$ and $y = b$.

   (a) Suppose the electromagnetic field can be written

   $$\mathbf{A}(x, y, z, t) = \sum_{\lambda=\pm1} \sum_{mn} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\sqrt{4\pi\omega_{nm}(k)}} \left[\boldsymbol{\epsilon}_{nm\lambda}(k)\, a_{knm\lambda}\, e^{-i\omega_{nm}(k)t + ikz} u_{knm}(x, y) + \text{c.c.} \right]$$

   then in a gauge for which the field equation becomes $(-\partial_t^2 + \nabla^2)\mathbf{A} = 0$ find the mode functions $u_{knm}(x, y)$ that satisfy the correct boundary conditions, and by doing so identify the range of values that can be taken by the quantum numbers k, n and m. Calculate $\omega_{nm}(k)$ as a function of k, n, m and a and b. (No need to compute the polarization vectors $\boldsymbol{\epsilon}_{nm\lambda}(k)$ or the normalization constant for the mode functions.)

   (b) Evaluate the zero-point (vacuum) energy (per unit length L in the z-direction)

   $$\xi_0(a, b) = \frac{E_0}{L} = \frac{1}{2} \sum_{\lambda=\pm1} \sum_{nm} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\, \omega_{nm}(k)\,.$$