1. Supposed a Schrödinger field $\Psi(\mathbf{x}, t)$ satisfies the field equation

$$iD_t\Psi = \left[-\frac{1}{2m}\,\mathbf{D}^2 + V(\mathbf{x})\right]\Psi$$

where

$$D_t\Psi = \Big[\partial_t + ie\phi(\mathbf{x}, t)\Big]\Psi \quad \text{and} \quad \mathbf{D}\Psi = \Big[\nabla - ie\mathbf{A}(\mathbf{x}, t)\Big]\Psi$$

are the covariant derivatives for Ψ, where ϕ and $\mathbf{A}$ are the scalar and vector potentials for the electromagnetic fields.

Prove – for any choice of V, ϕ and $\mathbf{A}$ – that this equation implies current conservation $\partial_t\rho + \nabla\cdot\mathbf{J} = 0$, where

$$\rho = -e\,\Psi^\star\Psi\,,$$

and e is a constant. What is the expression for the current $\mathbf{J}$ in terms of Ψ and $\mathbf{A}$? (This is very similar to an earlier assignment, but with the field $\mathbf{A}$ turned on.)

2. Part of the interaction Hamiltonian that couples the Schrödinger field Ψ (*e.g.* for the electron) to the electromagnetic potential $\mathbf{A}$ is the one given in the lectures:

$$H_{\rm int} = -\frac{ie}{2m}\int \mathrm{d}^3x\,\Big[\Psi^\star(\nabla\Psi) - (\nabla\Psi^\star)\Psi\Big]\cdot\mathbf{A}\,.$$

Use the standard expansion in terms of the creation and annihilation operators

$$\Psi(\mathbf{x}, t) = \sum_n c_n\,u_n(\mathbf{x})\,e^{-i\varepsilon_n t}$$

and

$$\mathbf{A}(\mathbf{x}) = \sum_\lambda\int\frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2|\mathbf{k}|}}\Big[\mathfrak{a}_{\mathbf{k}\lambda}\,\mathbf{e}_\lambda(\mathbf{k})\,e^{-i|\mathbf{k}|t + i\mathbf{k}\cdot\mathbf{x}} + \mathfrak{a}^\star_{\mathbf{k}\lambda}\,\mathbf{e}^*_\lambda(\mathbf{k})\,e^{i|\mathbf{k}|t - i\mathbf{k}\cdot\mathbf{x}}\Big]$$

where λ labels the photon polarization and $\mathbf{k}\cdot\mathbf{e}_\lambda(\mathbf{k}) = 0$ while the $u_n(\mathbf{x})$ satisfy

$$\left[-\frac{1}{2m}\,\nabla^2 + V(\mathbf{x})\right]u_n = \varepsilon_n\,u_n\,,$$

to identify the coefficient $\mathfrak{g}_{mn\lambda}(\mathbf{k})$ of the term involving $a^\star c^\star c$ in $H_{\rm int}$ of the form

$$\sum_\lambda\sum_{mn}\int \mathrm{d}^3k\;\mathfrak{g}_{mn\lambda}(\mathbf{k})\,a^\star_{\mathbf{k}\lambda}\,c^\star_m\,c_n \subset H_{\rm int}\,,$$

that describes photon emission (do not evaluate the integrals). Earlier in the term you evaluated the emission rate using this kind of expression where $\mathfrak{g}$ was a given function, and this now shows how this function is really determined for photon emission by electrons.