1. Consider the following Hamiltonian describing a complex scalar field $\Phi(x)$

$$\mathcal{H} = \partial_t \Phi^\star \partial_t \Phi + \nabla \Phi^\star \cdot \nabla \Phi + V(\Phi^\star \Phi)\,,$$

where

$$V(\Phi^\star \Phi) = \frac{\lambda}{4}\Big(\Phi^\star \Phi - v^2\Big)^2\,,$$

for real and positive parameters λ and v.

(a) Expand Φ around the minimum of the potential by writing

$$\Phi(x) = v + \frac{1}{\sqrt{2}}\Big[\psi(x) + i\chi(x)\Big]\,,$$

where $\psi(x)$ and $\chi(x)$ are both real fields. Write the Hamiltonian density as $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\rm int}$ where $\mathcal{H}_0$ contains terms that are quadratic or less in ψ and χ and $\mathcal{H}_{\rm int}$ has cubic and quartic terms. What are $\mathcal{H}_0$ and $\mathcal{H}_{\rm int}$?

(b) Expand ψ and χ in terms of creation and annihilation operators and verify that $H_0 = \int \mathrm{d}^3x\, \mathcal{H}_0$ has the form appropriate to free particles with relativistic dispersion relations $\varepsilon(p) = \sqrt{\mathbf{p}^2 + m^2}$. What is m (in terms of λ and v) for each of the fields ψ and χ?

(c) Calculate the scattering amplitude $\mathcal{A}$ for the scattering

$$\chi(\mathbf{p}) + \chi(\mathbf{q}) \to \chi(\mathbf{p}') + \chi(\mathbf{q}')\,.$$

Notice that this might involve a term at both first and second order in perturbation theory in powers of $\mathcal{H}_{\rm int}$. (BONUS) What is the leading form of this amplitude as the momenta of the particles involved all go to zero?