1. Maxwell's equations can be written in manifestly relativistic form if they are expressed in terms of the electromagnetic current 4-vector and the antisymmetric electromagnetic field-strength tensor, defined by

$$\begin{pmatrix} J^0 \\ J^x \\ J^y \\ J^z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{pmatrix} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix},$$

where J^i are the components of the electromagnetic current $\mathbf{J}$, while E_i and B_i are the components of the electric and magnetic fields, and j^0 is the electric charge density. 4-vectors and tensors with indices written as superscripts and subscripts are related using the Minkowski metric, with (for instance) $J^\mu = \eta^{\mu\nu} J_\nu$ and $F^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\beta}F_{\lambda\beta}$ and so on, where $\eta_{\mu\nu} = \text{diag}(-1,+1,+1,+1)$ and $\eta^{\mu\nu}$ is its inverse matrix (so $\eta^{\mu\nu}\eta_{\nu\lambda} = \delta^\mu_\lambda$). The Einstein summation convention is in force througout.

   (a) Show that the four Maxwell equations can be written in terms of these as

   $$\partial_\mu F^{\mu\nu} + J^\nu = 0 \quad \text{and} \quad \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0\,.$$

   (b) Show that there is a unique Lorentz-invariant quantity that is bilinear in $F_{\mu\nu}$ and has all of its indices contracted using only the metric $\eta_{\mu\nu}$, and show by doing so that the quantity $\mathbf{E}^2 - \mathbf{B}^2$ is a Lorentz scalar.

   (c) Show that the relationships $\mathbf{E} = -\partial_t \mathbf{A} - \nabla\phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$ relating electromagnetic fields and the electromagnetic potentials $\mathbf{A}$ and ϕ are given relativistically by

   $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

   and identify from this how the components of the electromagnetic potential A_μ are related to ϕ and $\mathbf{A}$.

   (d) In relativity the energy density is the time-time component of a symmetric tensor $\mathcal{H} = T_{00}$ where for electromagnetism

   $$T_{\mu\nu} = F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}\eta_{\mu\nu}\, F^{\lambda\rho}F_{\lambda\rho}$$

   Use this to verify that the energy density of an electromagnetic field is $\mathcal{H} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$.

2. Suppose a relativistic scalar field $\phi(x)$ is related to creation and annihiliation operators for its particle and antiparticle by

$$\phi(x) = \int \frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2E_p}} \Big[\mathfrak{a}_{\mathbf{p}}\, e^{ip\cdot x} + \bar{\mathfrak{a}}^\star_{\mathbf{p}}\, e^{-ip\cdot x}\Big] \tag{0.1}$$

where $p \cdot x := \eta_{\mu\nu}p^\mu x^\nu = -E_p\, t + \mathbf{p}\cdot\mathbf{x}$ and the momentum 4-vector has components $p^\mu = \{p^0 = E_p\,, \mathbf{p}\}$ and the position 4-vector is $x^\mu = \{x^0 = t, \mathbf{x}\}$. Here $E_p = \sqrt{\mathbf{p}^2 + m^2}$ is the relativistic single-particle energy as a function of momentum.

(a) Assuming that the particle and antiparticle are bosons, so $[\mathfrak{a}_{\mathbf{p}}, \mathfrak{a}^{\star}_{\mathbf{q}}] = \delta^3(\mathbf{p}-\mathbf{q})$ and $[\bar{\mathfrak{a}}_{\mathbf{p}}, \bar{\mathfrak{a}}^{\star}_{\mathbf{q}}] = \delta^3(\mathbf{p}-\mathbf{q})$ (with all other commutators, like $[\mathfrak{a}_{\mathbf{p}}, \bar{\mathfrak{a}}_{\mathbf{q}}] = 0$ and so on, vanishing) evaluate the commutator $[\phi(x)\,\phi^{\star}(y)]$ as a function of x^μ and y^ν and show in particular that it vanishes when $(x-y)^2 = \eta_{\mu\nu}(x-y)^\mu(x-y)^\nu > 0$ (so the separation between x^μ and y^μ is spacelike). Notice that this commutator would not vanish for spacelike separations if the antiparticle creation operator did not appear in (0.1).

(b) Repeat the previous calculation evaluating $[\phi(x), \phi^{\star}(y)]$ as a function of x^μ and y^ν, but this time assuming that the particle and antiparticle are fermions rather than bosons (so $\{\mathfrak{a}_{\mathbf{p}}, \mathfrak{a}^{\star}_{\mathbf{q}}\} = \delta^3(\mathbf{p}-\mathbf{q})$ and so on, with $\{\mathfrak{a}_{\mathbf{p}}, \bar{\mathfrak{a}}_{\mathbf{q}}\} = 0$ *etc.*. Verify in particular that $[\phi(x), \phi^{\star}(y)]$ does *not* vanish in this case when x^μ and y^ν are spacelike separated.

(c) Suppose the interaction Hamiltonian for this field is given by $\mathcal{H} = \lambda\,(\phi^{\star}\phi)^2$ where λ is a positive real coupling parameter. Show that $[\mathcal{H}(x)\,, \mathcal{H}(y)]$ vanishes for space-like separated points only if $[\phi(x), \phi^{\star}(y)]$ also vanishes for these separations. As discussed in class it turns out to be important that $[\mathcal{H}(x), \mathcal{H}(y)]$ vanish for space-like separated points since otherwise Lorentz invariance fails, and so the above calculations show why spinless particles (those appearing in scalar fields) must be bosons. This is a simple example of the spin-statistics theorem – that states that all integer-spin particles must be bosons and all half-integer spin particles must be fermions – that is a general consequence of merging special relativity with quantum mechanics.