1) $(-\partial_t^2 + \nabla^2) A = 0$

The only spacetime dependent parts of $\vec{A}$ are the exponential and mode functions.

So the field eq'ns reduce to

$$(-\partial_t^2 + \nabla^2) e^{-i\omega_{nm}(k)t + ikz} u_{knm}(x,y) = 0$$

$$\Rightarrow (\omega_{nm}(k)^2 - k^2) u_{knm}(x,y) + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u_{knm}(x,y) = 0$$

Separate $u_{knm}(x,y) = X(x) Y(y)$ then divide by u

$$\Rightarrow \omega_{nm}(k)^2 - k^2 + \frac{X''}{X} + \frac{Y''}{Y} = 0$$

Introduce separation variables q_n, q_m such that

$$\frac{X''}{X} = -q_n^2, \quad \frac{Y''}{Y} = -q_m^2.$$

Then $X(x) = Ae^{\pm i q_n x}$, $Y = Be^{\pm i q_m y}$

The boundary cond'ns force $X(0) = X(a) = 0 = Y(0) = Y(b)$

Then $X(x) = \sin\left(\frac{n\pi}{a} x\right)$, $Y = \sin\left(\frac{m\pi}{b} y\right)$ ignoring normalization.

Have identified $q_n = \frac{n\pi}{a}$, $q_m = \frac{m\pi}{b}$

$$\Rightarrow \omega_{nm}(k)^2 - k^2 - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2 = 0$$

This gives the dispersion relation

$$\omega_{nm}(k) = \sqrt{k^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

Here $k \in \mathbb{R}$ & $n, m \in \mathbb{Z}$ ie k is anything in $(-\infty, \infty)$ while n, m are integers.

b) $$\xi_0(a,b) = \frac{1}{2} \sum_{\lambda} \sum_{nm} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \omega_{nm}(k)$$

$$= \sum_{nm} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \omega_{nm}(k) \qquad \text{ind. of } \lambda.$$

Firstly, $\int_{-\infty}^{\infty} dx \sqrt{x^2 + a^2}$ is divergent, so need some regularization scheme.

Try the trick from lecture.

$$\int_0^{\infty} ds\, s^p e^{-\lambda s} = \lambda^{-1-p}\, \Gamma(1+p)$$

Let $$\omega_{nm}(k) = \sqrt{\lambda} = \lim_{p \to -3/2} \frac{1}{\Gamma(1+p)} \int_0^{\infty} ds\, s^p e^{-\omega^2 s}$$

Now our integral $$= \lim_{p \to -3/2} \frac{1}{\Gamma(1+p)} \sum_{nm} \int_0^{\infty} ds\, s^p \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-s\left(k^2 + \left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2\right)}$$

The k integral is gaussian, $\int dx\, e^{-ax^2} = \sqrt{\pi/a}$

$$\Rightarrow \xi_0 = \lim_{p \to -3/2} \frac{1}{\Gamma(1+p)} \sum_{nm} \frac{1}{2\sqrt{\pi}} \int_0^{\infty} ds\, s^{p-1/2} e^{-s\left(\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2\right)}$$

Can reuse the trick in reverse w/ $p = p - \frac{1}{2}$, $\lambda = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$

$$\Rightarrow \mathcal{E}_0(a,b) = \lim_{p \to -3/2} \frac{1}{2\sqrt{\pi}^3} \frac{\Gamma(\frac{1}{2}+p)}{\pi^{2p}\Gamma(1+p)} \sum_{nm} \frac{1}{\left[\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right]^{p+\frac{1}{2}}}$$

Now the double sum makes this tricky. In general, quite difficult to deal with this kind of thing.

So we stop here.