1) Important things to note here

$$D^2\psi = D\cdot D\psi \neq (D\cdot D)\psi$$

so $D^2\psi = (\nabla - ie\vec{A})\cdot(\nabla\psi - ie\vec{A}\psi)$

$$= \nabla^2\psi - ie\vec{A}\cdot\nabla\psi - ie\,\nabla\cdot(\vec{A}\psi) - e^2\vec{A}\cdot\vec{A}\psi$$

Also, the field equation does not hold in the given form for ψ^*.

so $iD_t\psi^* \neq \left[-\frac{1}{2m}D^2 + V\right]\psi^*$ because D_t, D are complex.

Look at $\partial_t\rho$ as given.

$$\partial_t\rho = \partial_t(-e\psi^*\psi) = -e(\partial_t\psi^*\psi + \psi^*\partial_t\psi)$$

Now $iD_t\psi = i(\partial_t + ie\phi)\psi = \left(-\frac{1}{2m}D^2 + V\right)\psi$

$$\Rightarrow \partial_t\psi = \left(\frac{i}{2m}D^2 - iV - ie\phi\right)\psi$$

$$= \frac{i}{2m}\left(\nabla^2\psi - e^2A^2\psi - ieA\cdot\nabla\psi - ie\nabla\cdot(A\psi)\right) - iV\psi - ie\phi\psi$$

and $(\partial_t\psi)^* = \partial_t\psi^* = \frac{-i}{2m}\left(\nabla^2\psi^* - e^2A^2\psi^* + ie(A\cdot\nabla\psi^* + \nabla\cdot(A\psi^*))\right) + iV\psi^* + ie\phi$

so $\partial_t\rho = \frac{ei}{2m}\Big(\nabla^2\psi^*\psi - \psi^*\nabla^2\psi + ie(A\cdot\nabla\psi^*)\psi + ie(A\cdot\nabla\psi)\psi^*$

$$+ ie\,\nabla\cdot(A\psi^*)\psi + ie\,\nabla\cdot(A\psi)\psi^*\Big)$$

Now note $\nabla\cdot(\nabla\psi^*\psi - \psi^*\nabla\psi) = \nabla^2\psi^*\psi - \psi^*\nabla^2\psi$

and $\nabla\cdot(A\psi) = (\nabla\cdot A)\psi + A\cdot\nabla\psi$ & $\nabla\cdot(\vec{A}\psi\psi^*) = \nabla\cdot\vec{A}\psi^*\psi + \vec{A}\nabla\cdot(\psi^*\psi)$

so can write this as a tota divergence term,

$$\frac{ei}{2m} \nabla \cdot \left(\nabla \psi^* \psi - \psi^* \nabla \psi + 2ie A \psi^* \psi \right)$$

$$\Rightarrow \partial_t \rho = \nabla \cdot \Big(\underbrace{\frac{ei}{2m} \left(\nabla \psi^* \psi - \psi^* \nabla \psi + 2ei A \psi^* \psi \right)}_{= -\vec{J}} \Big)$$

Notice that $\vec{J} = \frac{-ei}{2m} \left((D\psi^*)\psi - \psi^* (D\psi) \right)$

the usual current but with $\nabla \to D$

2) $\psi^* \nabla \psi - \nabla \psi^* \psi = \sum_{nm} c_n^* u_n^* e^{+i\varepsilon_n t} c_m \nabla u_m e^{-i\varepsilon_m t}$

$\qquad - c_n^* \nabla u_n^* e^{+i\varepsilon_n t} c_m u_m e^{-i\varepsilon_m t}$

We're interested in the term involving $a^* c^+ c$, so only look at second term of $\vec{A}$.

$$\Rightarrow \quad H_{int} = \frac{-ie}{2m} \sum_\lambda \sum_{nm} \int d^3x \int \frac{d^3k}{\sqrt{(2\pi)^3 2|k|}} \left(u_n \nabla u_m - \nabla u_n u_m\right) c_n^* c_m a_{k\lambda}^* e^{i(k+\varepsilon_n-\varepsilon_m)t} e^{-ikx}$$

$$\text{so } g_{mn\lambda}(k) = \frac{-ie\, e^{i(k+\varepsilon_n-\varepsilon_m)t}}{2m\sqrt{(2\pi)^3 2|k|}} \int d^3x \left(u_n \nabla u_m - \nabla u_n u_m\right) e^{-ikx}$$