1. Note: There was a typo in the assignment that made Γ have the wrong dimensions in eq. I'll solve the same problem as you and make the necessary changes at the end.

(a) $|\beta\rangle_0 = \bar{a}^*_{k\xi}\, a^*_{q\sigma} |0\rangle$

$|\alpha\rangle_0 = b_0^* |0\rangle$

$$ {}_0\langle\beta|H_{int}|\alpha\rangle_0 = \sum_{\sigma'\xi'} \frac{g_{\sigma'\xi'}}{\sqrt{8(2\pi)^9}} \int \frac{d^3p'\,d^3q'\,d^3k'}{\sqrt{\varepsilon_B(p')\,\varepsilon_F(q')\,\varepsilon_F(k')}} $$

$$ \langle 0|\, a_{q\sigma}\,\bar{a}_{k\xi} \left(b_{p'} a^*_{q'\sigma'} \bar{a}^*_{k'\xi'} + \bar{a}_{k'\xi'} a_{q'\sigma'} b^*_{p'} \right) \delta^3(\vec{p}'-\vec{q}'-\vec{k}')\, b_0^* |0\rangle \;\} $$

$$ = -\sum_{\sigma',\xi'} \frac{g_{\sigma',\xi'}}{\sqrt{8(2\pi)^9}} \int \frac{d^3p'\,d^3q'\,d^3k'}{\sqrt{\varepsilon_B(p')\,\varepsilon_F(q')\,\varepsilon_F(k')}} \left(\delta(\vec{p}')\,\delta(\vec{q}-\vec{q}')\,\delta(\vec{k}-\vec{k}') \right) \cdot \delta(\vec{p}'-\vec{q}'-\vec{k}')\,\delta_{\xi,\xi'}\,\delta_{\sigma,\sigma'} $$

$$ = -\frac{g_{\sigma\xi}}{\sqrt{8(2\pi)^9}} \frac{\delta(\vec{q}+\vec{k})}{\sqrt{M}\,(q^2+m^2)^{1/4}(k^2+m^2)^{1/4}} $$

$$ dP_t(\alpha\to\beta) = 4\,\frac{|g_{\sigma\xi}|^2}{8(2\pi)^9} \frac{\delta^3(\vec{q}+\vec{k})}{M\sqrt{(k^2+m^2)(q^2+m^2)}}\; \underbrace{\delta^3(0)}_{\frac{V}{(2\pi)^3}} $$

$$ \left(\frac{(2\pi)^3}{V}\right)^3 \cdot \underbrace{\frac{\sin^2 \frac{E_\beta-E_\alpha}{2}(t-t_0)}{(E_\beta-E_\alpha)^2}}_{\frac{(t-t_0)}{2}\cdot\pi\,\delta(E_\beta-E_\alpha)} \underbrace{\frac{dn}{dE_\beta}\,dE_\beta}_{= \frac{V}{(2\pi)^3} d^3q\, \frac{V}{(2\pi)^3} d^3k} $$

$$ \boxed{ \begin{aligned} d\Gamma(\alpha\to\beta) &= 2\pi\,\delta\!\left(M-\left(\sqrt{q^2+m^2}+\sqrt{k^2+m^2}\right)\right) \frac{|g_{\sigma\xi}|^2}{8(2\pi)^9} \frac{\delta(\vec{q}+\vec{k})\,d^3q\,d^3k}{M\sqrt{q^2+m^2}\sqrt{k^2+m^2}} \\ &= \delta(M-\varepsilon_F(q)-\varepsilon_F(k)) \frac{|g_{\sigma\xi}|^2}{8(2\pi)^8} \frac{\delta(\vec{q}+\vec{k})\,d^3q\,d^3k}{M\sqrt{q^2+m^2}\sqrt{k^2+m^2}} \end{aligned} } $$

(b) $\vec{q}+\vec{k}=0$ & $\sqrt{q^2+m^2}+\sqrt{k^2+m^2}=M$

$\hookrightarrow |\vec{k}|=|\vec{q}|$, $\varepsilon_F(q)=\varepsilon_F(k)=\frac{M}{2}$

(c)

$$ d\Gamma' = \sum_{\sigma,\xi} \delta(M-\varepsilon_F(q)-\varepsilon_F(k)) \frac{|g_0|^2\,\delta_{\sigma,\xi}}{8(2\pi)^8} \frac{\delta(\vec{q}+\vec{k})\,d^3q\,d^3k}{M\sqrt{q^2+m^2}\sqrt{k^2+m^2}} $$

$$ = \delta(M-\varepsilon_F(q)-\varepsilon_F(k)) \frac{|g_0|^2}{4(2\pi)^8} \frac{\delta(\vec{q}+\vec{k})\,d^3q\,d^3k}{M\sqrt{q^2+m^2}\sqrt{k^2+m^2}} $$

(d)

$$ \Gamma = \int \delta(M-\varepsilon_F(q)-\varepsilon_F(k)) \frac{|g_{\sigma\xi}|^2}{8(2\pi)^8} \frac{\delta(\vec{q}+\vec{k})\,d^3q\,d^3k}{M\sqrt{q^2+m^2}\sqrt{k^2+m^2}} $$

$$ = \frac{|g_{\sigma\xi}|^2}{8(2\pi)^8} \frac{1}{M} \int dq\, q^2\, d^2\Omega\, \frac{\delta(M-2\sqrt{q^2+m^2})}{q^2+m^2} $$

$$ = \frac{|g_{\sigma\xi}|^2}{4(2\pi)^7} \frac{1}{M} \int dq\, \frac{q^2}{q^2+m^2} \frac{\delta\left(q-\sqrt{\left(\frac{M}{2}\right)^2-m^2}\right)}{\frac{2q}{\sqrt{q^2+m^2}}\Big|_{q=\sqrt{\left(\frac{M}{2}\right)^2-m^2}}} $$

$$ = \frac{|g_{\sigma\xi}|^2}{4(2\pi)^7} \frac{1}{M} \frac{\sqrt{\left(\frac{M}{2}\right)^2-m^2}}{2\cdot\frac{M}{2}} $$

$$ \boxed{ \begin{aligned} \Gamma &= \frac{|g_{\sigma\xi}|^2}{4(2\pi)^7} \frac{\sqrt{\left(\frac{M}{2}\right)^2-m^2}}{M^2} \\ \Gamma' &= \frac{|g_0|^2}{2(2\pi)^7} \frac{\sqrt{\left(\frac{M}{2}\right)^2-m^2}}{M^2} \end{aligned} } $$

, if $g_{\sigma\xi} = g_0\,\delta_{\sigma,\xi}$ and we don't measure final spins

- for indistinguishable final state particles we divide the final result by 2 (or integrate over 4π instead of 2π).

(e) Plugging in the numerical values, we get:

$$ \Gamma' \approx 9.43\cdot 10^{-25}\ (\text{eV})^{-1} \longrightarrow \text{wrong units!} $$

The text of the assignment is now corrected so that the units make sense. We should replace $\frac{1}{\sqrt{\varepsilon_B(p)\,\varepsilon_F(q)\,\varepsilon_F(k)}}$ in the definition of H_{int} by $\frac{1}{\sqrt{\varepsilon_B(p)}}$. We can fix Γ' by multiplying with $\left(\sqrt{\varepsilon_F(q)\,\varepsilon_F(k)}\right)^2\Big|_{q=k=\sqrt{\left(\frac{M}{2}\right)^2-m^2}}$

$$ \Gamma'_{new} = \frac{|g_0|^2}{2(2\pi)^7} \frac{\sqrt{\left(\frac{M}{2}\right)^2-m^2}}{\cancel{M^2}} \left(\frac{\cancel{M}}{2}\right)^2 $$

$$ \approx 0.004\ \text{eV} $$

2. (a)

$$\langle k, -k | S | 0 \rangle$$

$$= \langle 0 | a_{-k} a_k \left(1 - i \int_{-\infty}^{\infty} d\tau \, H_{int,I}(\tau)\right) | 0 \rangle$$

$$= -i \int_{-\infty}^{\infty} d\tau \, \langle 0 | a_{-k} a_k \left(\int d^3p \, V(\tau) \left(a_p a_{-p} + a_p^* a_{-p}^* \right) \right) \cdot e^{i(E_\beta - E_\alpha)\tau} | 0 \rangle$$

$$= -i \int_{-\infty}^{\infty} d\tau \int d^3p \, V(\tau) \left(\delta(\vec{p} - \vec{k}) \, \delta(\vec{p} - \vec{k}) + \delta(\vec{k} + \vec{p}) \, \delta(\vec{k} + \vec{p}) \right) e^{2i|\vec{k}|t}$$

$$= -2i \int_{-\infty}^{\infty} d\tau \, V(\tau) \, \delta^3(\vec{0}) \, e^{i2|\vec{k}|\tau}$$

$$= -2i \, \delta^3(\vec{0}) \int_{-\infty}^{\infty} d\tau \, V_0 \cos(\Omega\tau) \, e^{i2|\vec{k}|\tau}$$

$$= -2i \delta^3(\vec{0}) \int_{-\infty}^{\infty} d\tau \, V_0 \left(\frac{e^{i(\Omega + 2|\vec{k}|)\tau} + e^{-i(\Omega - 2|\vec{k}|)\tau}}{2} \right)$$

$$= -2\pi i \delta^{(3)}(\vec{0}) V_0 \left(\delta(\Omega + 2|\vec{k}|) + \delta(\Omega - 2|\vec{k}|) \right)$$

(b) Since $\Omega > 0$, the minimum value of Ω for the above process is $\Omega = 2|\vec{k}|$. However, $\vec{k}$ is an arbitrary momentum and so the min. of $|\vec{k}|$ is 0. We get $\Omega > 0$.