1. The hamiltonian is:

I'm replacing m with M so as not to confuse the mass and the index m

$$H = \int_{-\infty}^{\infty} dx\,dy\,dz \left(\frac{1}{2M} \nabla \psi^* \cdot \nabla \psi + V(\vec{x})\, \psi^* \psi \right)$$

with $V(\vec{x}) = \frac{1}{2} M w^2 (x^2 + y^2)$. We can rewrite the kinetic term in a more convenient way:

$$\int d^3x\, \partial_i \psi^* \partial^i \psi = \int d^3x \left(\partial_i (\psi^* \partial^i \psi) - \psi^* \partial^i \partial_i \psi \right)$$

$$= \underbrace{\int d^2 s\, n_i\, \psi^* \partial^i \psi}_{} - \int d^3x\, \psi^* \partial^i \partial_i \psi$$

This is a boundary term, which we typically neglect by assuming ψ vanishes at the $|\vec{x}| \to \infty$ boundary

Using this form of the kinetic term, and the given expansion of ψ, we can rewrite H as:

$$H = \int_{-\infty}^{\infty} dx\,dy\,dz \sum_{m,n,m',n'} \int \frac{dk\,dk'}{2\pi} (a^*_{kmn}\, a_{k'm'n'})$$
$$\Big\{ -\frac{1}{2M} u^*_{mn}(x,y)\, e^{i(w_{mn}(k)t - kz)}\, \Delta \left(u_{m'n'}(x,y)\, e^{-i(w_{m'n'}(k')t - k'z)} \right)$$
$$+ V(x)\, u^*_{mn}(k)\, u_{m'n'}(k')\, e^{i(w_{mn}(k) - w_{m'n'}(k'))t} \cdot e^{-i(k-k')z} \Big\}$$

Upon quantization, we replace the product of coefficients $a^*_{kmn} a_{k'm'n'}$ with the symmetric product of operators

$$(a^*_{kmn}\, a_{k'm'n'}) \equiv \frac{1}{2} (a^*_{kmn}\, a_{k'm'n'} + a_{k'm'n'}\, a^*_{kmn})$$

$$= a^*_{kmn}\, a_{k'm'n'} + \frac{1}{2} \delta_{mm'} \delta_{nn'} \delta(k-k').$$

Note also that:

$$\Delta \left(u_{m'n'}(x,y)\, e^{-i(w_{m'n'}(k')t - k'z)} \right)$$
$$= e^{-i(w_{m'n'}(k)t - k'z)} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k'^2 \right) u_{m'n'}(x,y)$$

Using this in H gives:

$$H = \int_{-\infty}^{\infty} dx\,dy\,dz \sum_{\substack{m,n,\\ m',n'}} \int \frac{dk\,dk'}{2\pi} \left(a^*_{kmn}\, a_{k'm'n'} + \frac{1}{2} \delta_{mm'} \delta_{nn'} \delta(k-k') \right)$$
$$u^*_{mn}(x,y)\, e^{i(w_{mn}(k) - w_{m'n'}(k'))t}\, e^{-i(k-k')z}$$
$$\underbrace{\left\{ -\frac{1}{2M} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + V(\vec{x}) + \frac{k'^2}{2M} \right\} u_{m'n'}(x,y)}_{= \left(E_{m'n'} + \frac{k'^2}{2M} \right) u_{m'n'}(x,y)}$$

$$= \sum_{m,n} \underbrace{\int_{-\infty}^{\infty} dx\,dy\, u^*_{mn} u_{mn}}_{= \delta_{m,m} \delta_{n,n} = 1} \int_{-\infty}^{\infty} dz \int \frac{dk}{2\pi}\, \frac{1}{2} \left(E_{mn} + \frac{k^2}{2M} \right)$$

$$+ \Big\{ \sum_{\substack{m,n,\\ m',n'}} \int \frac{dk\,dk'}{2\pi} \Big(\underbrace{\int_{-\infty}^{\infty} dz\, e^{-i(k-k')z}}_{= 2\pi \delta(k-k')} \Big) e^{i(w_{mn}(k) - w_{m'n'}(k'))t}$$
$$\Big(\underbrace{\int_{-\infty}^{\infty} dx\,dy\, u^*_{mn}(x,y)\, u_{m'n'}(x,y)}_{= \delta_{mm'} \delta_{nn'}} \Big) \left(E_{m'n'} + \frac{k'^2}{2M} \right) \cdot a^*_{kmn}\, a_{k'm'n'} \Big\}$$

$$= \sum_{m,n} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{1}{2} \left(E_{mn} + \frac{k^2}{2M} \right)$$

$$+ \sum_{m,n} \int dk\, a^*_{kmn}\, a_{kmn} \left(E_{mn} + \frac{k^2}{2M} \right)$$

$$\boxed{= E_0 + \sum_{m,n} \int dk\, a^*_{kmn}\, a_{kmn} \left(E_{mn} + \frac{k^2}{2M} \right)}$$

This is the desired form of H.

b) The single particle energies can be found from

$$H\, a^*_{kmn} |0\rangle$$

$$= \left(E_0 + \sum_{m',n'} \int dk'\, a^*_{k'm'n'}\, a_{k'm'n'} \left(E_{m'n'} + \frac{k'^2}{2M} \right) \right) a^*_{kmn} |0\rangle$$

$$= \left(E_0 + \left(E_{mn} + \frac{k^2}{2M} \right) \right) |0\rangle$$

where $E_0 = \langle 0 | H | 0 \rangle$. Relative to the vacuum, the single particle energy eigenvalues are

$$\boxed{w_{mn}(k) = E_{mn} + \frac{k^2}{2M}}$$

Here E_{mn} is defined through

$$\left(-\frac{1}{2M} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + V(\vec{x}) \right) u_{mn}(x,y) = E_{mn}\, u_{mn}(x,y)$$

where $V(\vec{x}) = \frac{1}{2} M w^2 (x^2 + y^2)$. Since this has the same form as the time independent Schrödinger eq. in a 2d harmonic oscillator potential (w. oscillator frequency w), we conclude that

$$\boxed{E_{mn} = (m + n + 1)\, w}, \quad m \geq 0,\ n \geq 0$$

c) The ground state is the vacuum, $|0\rangle$, whose energy is E_0 since:

$$H |0\rangle = E_0 |0\rangle$$

From (a) we know that

$$E_0 = \sum_{m,n} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{1}{2} \left(E_{mn} + \frac{k^2}{2M} \right)$$

$$= \lim_{L \to \infty} L \left(\sum_{m,n} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{1}{2} \left(E_{mn} + \frac{k^2}{2M} \right) \right)$$

where L is the length in the z direction. The energy per unit length is clearly

$$\boxed{\rho_0 = \lim_{L \to \infty} \frac{E_0}{L} = \sum_{m,n} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{1}{2} \underbrace{\left(E_{mn} + \frac{k^2}{2M} \right)}_{= w_{mn}(k)}}$$

To calculate ρ_0, we write:

$$w_{mn}(k) = \lambda = \lim_{p \to -2} \frac{1}{\Gamma(p+1)} \int_0^{\infty} ds\, s^p\, e^{-w_{mn}(k) s}$$

we get:

Gaussian integral

$$\rho_0 = \lim_{p \to -2} \frac{1}{2\Gamma(p+1)} \sum_{m,n} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_0^{\infty} ds\, s^p\, e^{-\frac{s k^2}{2M}}\, e^{-s \cdot E_{mn}}$$

$$= \lim_{p \to -2} \frac{1}{2\Gamma(p+1)} \sum_{m,n} \int_0^{\infty} ds\, s^{p-\frac{1}{2}}\, e^{-s(m+n+1)w} \sqrt{\frac{M}{2\pi}}$$

$$= \lim_{p \to -2} \sqrt{\frac{M}{2\pi}}\, \frac{1}{2\Gamma(p+1)} \int_0^{\infty} ds\, s^{p-\frac{1}{2}} e^{-sw} \underbrace{\sum_m (e^{-sw})^m}_{\frac{1}{1-e^{-sw}}} \underbrace{\sum_n (e^{-sw})^n}_{\frac{1}{1-e^{-sw}}}$$

$$= \lim_{p \to -2} \frac{\sqrt{\frac{M}{2\pi}}}{2\Gamma(p+1)} \int_0^{\infty} ds\, s^{p-\frac{1}{2}} \frac{e^{-sw}}{(1-e^{-sw})^2}$$

$$= \lim_{p \to -2} \frac{\sqrt{\frac{M}{2\pi}}}{2\Gamma(p+1)} \int_0^{\infty} ds\, \frac{s^{p-\frac{1}{2}}\, e^{sw}}{(e^{sw}-1)^2}$$

The integral we're given for $n = p - \frac{1}{2}$

$$= \lim_{p \to -2} \frac{\sqrt{\frac{M}{2\pi}}}{2\Gamma(p+1)}\, \Gamma\!\left(p + \tfrac{1}{2}\right) \frac{\zeta(p - \frac{1}{2})}{w^{p+\frac{1}{2}}}$$

$$= \frac{1}{2} \sqrt{\frac{M}{2\pi}} \lim_{p \to -2} \left(\frac{\Gamma(p + \frac{1}{2})}{\Gamma(p+1)}\, \frac{\zeta(p - \frac{1}{2})}{w^{p+\frac{1}{2}}} \right) = 0$$

finite

tends to $\frac{4\sqrt{\pi}}{3}$

$\Gamma(-1)$ is divergent, but we're dividing by it!

$$\boxed{\rho_0 = 0}$$