1. First, we calculate $F^{\mu\nu}$:

$$F^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\beta} F_{\lambda\beta} = \eta^{\mu\lambda} F_{\lambda\beta} (\eta^T)^{\beta\nu}$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

- Note that $F^{0i} = E_i = -F_{0i}$, $F^{\underset{\sim}{ij}} = \epsilon^{ijk} B_k$ if $\epsilon^{123} = +1$ (spatial indices)

(a) For $\nu = 0$, $\partial_\mu F^{\mu\nu} + J^\nu = 0$ becomes:

$$\partial_\mu F^{\mu 0} + J^0$$
$$= \underbrace{\partial_0 F^{00}}_{=0} + \partial_i \underbrace{F^{i0}}_{-E_i} + J^0$$
$$= -\partial_i E_i + \underbrace{J^0}_{=\rho} = \boxed{-\operatorname{div}\vec{E} + \rho = 0}$$

- For $\nu = i$ (a spatial index):

$$\partial_\mu F^{\mu i} + J^i \quad \text{spatial index}$$
$$= \partial_0 F^{0i} + \partial_j F^{ji} + J^i$$
$$= \partial_0 E_i + \partial_j (\epsilon^{jik} B_k) + J^i$$
$$= \partial_0 E_i - (\operatorname{rot} B)_i + J^i = \boxed{(\partial_t \vec{E} - \operatorname{rot}\vec{B} + \vec{J})_i = 0}$$

- $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$ becomes:

  - For $\mu = i$, $\nu = j$, $\lambda = 0$ (spatial indices)

$$\partial_i F_{j0} + \partial_j F_{0i} + \partial_0 F_{ij}$$
$$= \partial_i E_j - \partial_j E_i + \partial_0 \epsilon_{ijk} B_k = 0$$

multiply by $\epsilon^{ijl}/2$

$$\frac{1}{2}\epsilon^{ijl} (\partial_i E_j - \partial_j E_i + \epsilon_{ijk} \partial_0 B_k)$$
$$= \boxed{(\operatorname{rot}\vec{E} + \partial_t \vec{B})_l = 0}$$

- For $\mu = 1, \nu = 2, \lambda = 3$:

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12}$$
$$= \partial_x B_x + \partial_y B_y + \partial_z B_z = \boxed{\operatorname{div}\vec{B} = 0}$$

- For other values of μ, λ, ν it can be shown that $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu}$ vanishes trivially.

b) From the definition of $T_{\mu\nu}$, it follows that for $\mu = \nu = 0$:

$$T_{00} = F_{0\lambda} F_0{}^\lambda - \frac{1}{4}\eta_{00} F^{\lambda\rho} \underbrace{F_{\lambda\rho}}_{= -F_{\rho\lambda}}$$

the two terms in T_{00} are:

$$F^{\lambda\rho} F_{\lambda\rho} = -F^{\lambda\rho} F_{\rho\lambda} = -\operatorname{Tr}\left[\begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \right]$$

$$= +2\,(\vec{B}^2 - \vec{E}^2)$$

$$F_{0\lambda} F^{0\lambda} = F_{0\lambda}\eta^{\lambda\rho} F_{0\rho} = \eta^{\lambda\rho} E_\lambda E_\rho = \vec{E}\cdot\vec{E} = \vec{E}^2$$

It follows that:

$$T_{00} = \vec{E}^2 - \frac{1}{4}(-1)(\vec{B}^2 - \vec{E}^2)\cdot 2$$
$$= \vec{E}^2 + \frac{1}{2}(\vec{B}^2 - \vec{E}^2)$$
$$\boxed{= \frac{1}{2}(\vec{E}^2 + \vec{B}^2)} = \mathcal{H}$$

2.

$$H = \int d^3x \left[\frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \phi^* \left(-\frac{1}{2m}\nabla^2\right)\phi + \frac{e^2}{2m}\phi^*\phi\, \vec{A}\cdot\vec{A} \right]$$

We can expand the fields as:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\, c_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}}$$

$$\vec{A}(\vec{x}) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2|\vec{k}|}} \sum_\lambda \left(a_{\vec{k}\lambda}\vec{\epsilon}_\lambda(\vec{k}) e^{i\vec{k}\cdot\vec{x}} + a^*_{\vec{k}\lambda}\vec{\epsilon}^*_\lambda(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} \right)$$

. The interaction hamiltonian is

$$H_{int} = \frac{e^2}{2m}\int d^3x \int \frac{d^3q\, d^3q'}{(2\pi)^3}\, c^*_{\vec{q}} c_{\vec{q}'}\, e^{-i(\vec{q}-\vec{q}')\cdot\vec{x}}$$
$$\sum_{\mu,\mu'} \int \frac{d^3l\, d^3l'}{2(2\pi)^3\sqrt{|\vec{l}||\vec{l}'|}} \left(a_{\vec{l}\mu}\vec{\epsilon}_\mu(\vec{l})e^{i\vec{l}\cdot\vec{x}} + a^*_{\vec{l}\mu}\vec{\epsilon}^*_\mu(\vec{l})e^{-i\vec{l}\cdot\vec{x}}\right)$$
$$\left(a_{\vec{l}'\mu'}\vec{\epsilon}_{\mu'}(\vec{l}')e^{i\vec{l}'\cdot\vec{x}} + a^*_{\vec{l}'\mu'}\vec{\epsilon}^*_{\mu'}(\vec{l}')e^{-i\vec{l}'\cdot\vec{x}}\right)$$

$$= \frac{e^2}{2m}\int \frac{d^3q\, d^3q'}{(2\pi)^3} \sum_{\mu,\mu'} \int \frac{d^3l\, d^3l'}{2(2\pi)^3\sqrt{|\vec{l}||\vec{l}'|}}\, c^*_{\vec{q}} c_{\vec{q}'}$$
$$\Big(a_{\vec{l}\mu}a_{\vec{l}'\mu'}\,\vec{\epsilon}_\mu(\vec{l})\cdot\vec{\epsilon}_{\mu'}(\vec{l}') \underbrace{\int d^3x\, e^{i(\vec{l}+\vec{l}'-\vec{q}+\vec{q}')\cdot\vec{x}}}_{=(2\pi)^3\delta(\vec{l}+\vec{l}'+\vec{q}'-\vec{q})}$$
$$+ a_{\vec{l}\mu}a^*_{\vec{l}'\mu'}\,\vec{\epsilon}_\mu(\vec{l})\cdot\vec{\epsilon}^*_{\mu'}(\vec{l}') \underbrace{\int d^3x\, e^{i(\vec{l}-\vec{l}'-\vec{q}+\vec{q}')\cdot\vec{x}}}_{=(2\pi)^3\delta(\vec{l}-\vec{l}'-\vec{q}+\vec{q}')}$$
$$+ a^*_{\vec{l}\mu}a_{\vec{l}'\mu'}\,\vec{\epsilon}^*_\mu(\vec{l})\cdot\vec{\epsilon}_{\mu'}(\vec{l}') \underbrace{\int d^3x\, e^{i(\vec{l}'-\vec{l}-\vec{q}+\vec{q}')\cdot\vec{x}}}_{=(2\pi)^3\delta(\vec{l}'-\vec{l}-\vec{q}+\vec{q}')}$$
$$+ a^*_{\vec{l}\mu}a^*_{\vec{l}'\mu'}\,\vec{\epsilon}^*_\mu(\vec{l})\cdot\vec{\epsilon}^*_{\mu'}(\vec{l}') \underbrace{\int d^3x\, e^{i(-\vec{l}-\vec{l}'-\vec{q}+\vec{q}')\cdot\vec{x}}}_{=(2\pi)^3\delta(\vec{l}+\vec{l}'+\vec{q}-\vec{q}')} \Big)$$

$$= \frac{e^2}{2m}\int d^3q \sum_{\mu,\mu'}\int \frac{d^3l\, d^3l'}{2(2\pi)^3\sqrt{|\vec{l}||\vec{l}'|}}$$
$$\Big(c^*_{\vec{q}} c_{\vec{q}-(\vec{l}+\vec{l}')}\, a_{\vec{l}\mu}a_{\vec{l}'\mu'}\,\vec{\epsilon}_\mu(\vec{l})\cdot\vec{\epsilon}_{\mu'}(\vec{l}')$$
$$+ c^*_{\vec{q}} c_{\vec{q}+\vec{l}'-\vec{l}}\, a_{\vec{l}\mu}a^*_{\vec{l}'\mu'}\,\vec{\epsilon}_\mu(\vec{l})\cdot\vec{\epsilon}^*_{\mu'}(\vec{l}')$$
$$+ c^*_{\vec{q}} c_{\vec{q}+\vec{l}-\vec{l}'}\, a^*_{\vec{l}\mu}a_{\vec{l}'\mu'}\,\vec{\epsilon}^*_\mu(\vec{l})\cdot\vec{\epsilon}_{\mu'}(\vec{l}')$$
$$+ c^*_{\vec{q}} c_{\vec{q}+\vec{l}+\vec{l}'}\, a^*_{\vec{l}\mu}a^*_{\vec{l}'\mu'}\,\vec{\epsilon}^*_\mu(\vec{l})\cdot\vec{\epsilon}^*_{\mu'}(\vec{l}') \Big)$$

. The terms that contribute to the desired amplitude are (we need the same # of a, a^* operators):

$$\langle \vec{k}',\lambda';\vec{p}'|\, c^*_{\vec{q}} c_{\vec{q}+\vec{l}'-\vec{l}}\, a_{\vec{l}\mu}a^*_{\vec{l}'\mu'}\,|\vec{p};\vec{k},\lambda\rangle$$
$$= \langle 0|\, a_{\vec{k}'\lambda'} c_{\vec{p}'} c^*_{\vec{q}} c_{\vec{q}+\vec{l}'-\vec{l}}\, a_{\vec{l}\mu} a^*_{\vec{l}'\mu'} c^*_{\vec{p}} a^*_{\vec{k}\lambda}\,|0\rangle$$
$$= \delta(\vec{p}'-\vec{q})\,\delta(\vec{q}+\vec{l}'-\vec{l}-\vec{p})\,\delta(\vec{l}-\vec{l}')\,\delta_{\mu\mu'}\,\delta(\vec{k}-\vec{k}')\,\delta_{\lambda\lambda'}$$
$$+ \delta(\vec{p}'-\vec{q})\,\delta(\vec{q}+\vec{l}'-\vec{l}-\vec{p})\,\delta(\vec{l}-\vec{k})\,\delta_{\mu,\lambda}\,\delta(\vec{l}'-\vec{k}')\,\delta_{\lambda'\mu'}$$

$$\langle \vec{k}',\lambda';\vec{p}'|\, c^*_{\vec{q}} c_{\vec{q}+\vec{l}-\vec{l}'}\, a^*_{\vec{l}\mu}a_{\vec{l}'\mu'}\,|\vec{p};\vec{k},\lambda\rangle$$
$$= \langle 0|\, a_{\vec{k}'\lambda'} c_{\vec{p}'} c^*_{\vec{q}} c_{\vec{q}+\vec{l}-\vec{l}'}\, a^*_{\vec{l}\mu} a_{\vec{l}'\mu'} c^*_{\vec{p}} a^*_{\vec{k}\lambda}\,|0\rangle$$
$$= \delta(\vec{p}'-\vec{q})\,\delta(\vec{l}-\vec{k}')\,\delta_{\lambda'\mu}\,\delta(\vec{q}+\vec{l}-\vec{l}'-\vec{p})\,\delta(\vec{k}-\vec{l}')\,\delta_{\lambda,\mu'}$$

. Putting everything together we get:

$$\langle \vec{k};\lambda';\vec{p}'|H_{int}|\vec{p};\vec{k},\lambda\rangle$$
$$= \frac{e^2}{2m}\int d^3q \sum_{\mu,\mu'}\int \frac{d^3l\, d^3l'}{2(2\pi)^3\sqrt{|\vec{l}||\vec{l}'|}}$$
$$\Big\{ \vec{\epsilon}_\mu(\vec{l})\cdot\vec{\epsilon}^*_{\mu'}(\vec{l}')$$
$$\big(\delta(\vec{p}'-\vec{q})\,\delta(\vec{q}+\vec{l}'-\vec{l}-\vec{p})\,\delta(\vec{l}-\vec{l}')\,\delta_{\mu\mu'}\,\delta(\vec{k}-\vec{k}')\,\delta_{\lambda\lambda'}$$
$$+ \delta(\vec{p}'-\vec{q})\,\delta(\vec{q}+\vec{l}'-\vec{l}-\vec{p})\,\delta(\vec{l}-\vec{k})\,\delta_{\mu,\lambda}\,\delta(\vec{l}'-\vec{k}')\,\delta_{\lambda'\mu'}\big)$$
$$+ \vec{\epsilon}^*_\mu(\vec{l})\cdot\vec{\epsilon}_{\mu'}(\vec{l}')$$
$$\cdot\delta(\vec{p}'-\vec{q})\,\delta(\vec{l}-\vec{k}')\,\delta_{\lambda'\mu}\,\delta(\vec{q}+\vec{l}-\vec{l}'-\vec{p})\,\delta(\vec{k}-\vec{l}')\,\delta_{\lambda,\mu'}\Big\}$$

$$= \frac{e^2}{2m}\frac{1}{2(2\pi)^3}\Big\{ \frac{\vec{\epsilon}^*_{\lambda'}(\vec{k}')\cdot\vec{\epsilon}_\lambda(\vec{k})}{\sqrt{|\vec{k}||\vec{k}'|}}\,\delta(\vec{p}'+\vec{k}'-\vec{k}-\vec{p})$$
$$+ \frac{\vec{\epsilon}_\lambda(\vec{k})\cdot\vec{\epsilon}^*_{\lambda'}(\vec{k}')}{\sqrt{|\vec{k}||\vec{k}'|}}\,\delta(\vec{p}'+\vec{k}'-\vec{k}-\vec{p})$$
$$+ \underbrace{\int d^3l \sum_\mu \frac{\vec{\epsilon}_\mu(\vec{l})\cdot\vec{\epsilon}^*_\mu(\vec{l})}{|\vec{l}|}\,\delta(\vec{p}'-\vec{p})\,\delta(\vec{k}'-\vec{k})\,\delta_{\lambda,\lambda'}}_{} \Big\}$$

↳ this describes a process where "nothing happens" to either e or γ – we can neglect it since it doesn't contribute to the scattering process.

. Disregarding the last term, we get:

$$\langle \vec{k};\lambda';\vec{p}'|H_{int}|\vec{p};\vec{k},\lambda\rangle$$
$$= \frac{e^2}{2m}\frac{\vec{\epsilon}^*_{\lambda'}(\vec{k}')\cdot\vec{\epsilon}_\lambda(\vec{k})}{(2\pi)^3\sqrt{|\vec{k}||\vec{k}'|}}\,\delta(\vec{k}'+\vec{p}'-\vec{k}-\vec{p})$$

The differential rate for this process is:

this factor accounts for the normalization of states

$$d\Gamma_{\lambda'} = |\langle \vec{k}',\lambda';\vec{p}'|H_{int}|\vec{p};\vec{k},\lambda\rangle|^2\, 2\pi\delta(|\vec{k}'|+E_{p'}-|\vec{k}|-E_p)$$
$$\left(\frac{(2\pi)^3}{V}\right)^{2}\frac{d^3k'}{\frac{(2\pi)^3}{V}}\frac{d^3p'}{\frac{(2\pi)^3}{V}}$$

$$= \left(\frac{e^2}{2m}\right)^2 \frac{|\vec{\epsilon}^*_{\lambda'}(\vec{k}')\cdot\vec{\epsilon}_\lambda(\vec{k})|^2}{(2\pi)^{6}|\vec{k}||\vec{k}'|}\,\delta(\vec{k}'+\vec{p}'-\vec{k}-\vec{p})\,\underbrace{\delta(0)}_{=\frac{V}{(2\pi)^3}}$$
$$2\pi\,\delta(|\vec{k}'|+E_{p'}-|\vec{k}|-E_p)$$
$$\left(\frac{(2\pi)^3}{V}\right)^{2} d^3k'\, d^3p'$$

. Integrating over d^3p' gives

$$d\Gamma'_{\lambda'} = \left(\frac{e^2}{4\pi m}\right)^2 \frac{1}{V}\, |\vec{\epsilon}^*_{\lambda'}(\vec{k}')\cdot\vec{\epsilon}_\lambda(\vec{k})|^2\, \frac{1}{|\vec{k}||\vec{k}'|}$$
$$\cdot\,\delta(|\vec{k}|-|\vec{k}'|-(E_{\vec{p}}-E_{\vec{p}+\vec{k}-\vec{k}'}))\, d^3k'$$

. We're interested in energies for which $|E_p-E_{p'}| \ll ||\vec{k}|-|\vec{k}'||$ so that the energy δ function is approximately $\delta(|\vec{k}|-|\vec{k}'|)$. Integrating over dk' gives

$$d\Gamma''_{\lambda'} = \left(\frac{e^2}{4\pi m}\right)^2 \frac{1}{V}\left(\int_0^\infty k'^2 dk'\, \overbrace{\delta(\sqrt{k'^2}-\sqrt{k^2})}^{=\delta(k'-k)} \cdot \frac{1}{|\vec{k}||\vec{k}'|}\,|\vec{\epsilon}^*_{\lambda'}(\vec{k}')\cdot\vec{\epsilon}_\lambda(\vec{k})|^2\right) d\Omega$$
$$= \left(\frac{e^2}{4\pi m}\right)^2 \frac{1}{V}\, |\vec{\epsilon}^*_{\lambda'}(\vec{k})\cdot\vec{\epsilon}_\lambda(\vec{k})|^2\, d\Omega$$

The differential cross section is then

$$\frac{d\sigma_{\lambda'}}{d\Omega} = \frac{d\Gamma''_{\lambda'}\cdot V}{N}$$

N = no. of initial electrons = 1

$$= \frac{e^4}{16\pi^2 m^2}\,|\vec{\epsilon}^*_{\lambda'}(\vec{k})\cdot\vec{\epsilon}_\lambda(\vec{k})|^2$$

. If $\lambda' = \lambda$, $|\vec{\epsilon}^*_\lambda(\vec{k})\cdot\vec{\epsilon}_\lambda(k)|^2 = \cos^2\frac{\theta}{2}$

$\lambda' = -\lambda$, $|\vec{\epsilon}^*_{-\lambda}(\vec{k})\cdot\vec{\epsilon}_\lambda(k)|^2 = -\sin^2\frac{\theta}{2}$

Finally, we get:

$$\boxed{\begin{aligned} \frac{d\sigma_+}{d\Omega} &= \frac{e^4}{16\pi^2 m^2}\cos^4\frac{\theta}{2}, \quad \lambda=\lambda' \\ \frac{d\sigma_-}{d\Omega} &= \frac{e^4}{16\pi^2 m^2}\sin^4\frac{\theta}{2}, \quad \lambda=-\lambda' \end{aligned}}$$

3.

(a)

$$[\phi(x), \phi^*(y)] = \int \frac{d^3p\, d^3q}{(2\pi)^3 2\sqrt{E_p E_q}} e^{i(px-qy)} \underbrace{[a_{\vec p}, a_{\vec q}^*]}_{= \delta^3(\vec p - \vec q)}$$

this is Lorentz invariant

$$= \int \frac{d^3p}{(2\pi)^3 2E_{\vec p}} e^{ip_\mu(x^\mu - y^\mu)}$$

This commutator is a Lorentz invariant function of $x^\mu - y^\mu$. The Lorentz scalars we can write using only $\eta_{\mu\nu}$, $x^\mu - y^\mu$ are functions of

$\eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu) = (x-y)^2$ so $[\phi(x), \phi^*(y)]$ is a function of $(x-y)^2$.

We now evaluate $[\phi(x), \phi^*(y)]$ for spacelike separations. We can switch to a ref. frame where $x^0 = y^0$; then:

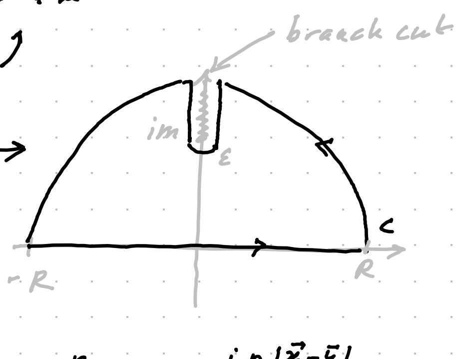
$$[\phi(x), \phi^*(y)] = \int \frac{d^3p}{(2\pi)^3 2E_{\vec p}} e^{i\vec p\cdot(\vec x - \vec y)}$$

$$= \int \frac{dp\, p^2\, d^2\Omega}{(2\pi)^3 2\sqrt{p^2+m^2}} e^{ip|\vec x - \vec y|\cos\theta}$$

$$= \frac{4\pi}{2(2\pi)^3|\vec x - \vec y|} \int_0^\infty \frac{dp\, p^2 \left(e^{ip|\vec x - \vec y|} - e^{-ip|\vec x - \vec y|}\right)}{2ip\cdot\sqrt{p^2+m^2}}$$

$$= \frac{4\pi}{4i(2\pi)^3|\vec x - \vec y|} \int_{-\infty}^{\infty} dp \frac{p}{\sqrt{p^2+m^2}} e^{ip|\vec x - \vec y|}$$

We can calculate this using the contour



$$\int_C dz\, f(z) = 0 = \int_{-R}^{R} dp \frac{p}{\sqrt{p^2+m^2}} e^{ip|\vec x - \vec y|}$$

$$\underbrace{+ i\int_0^{\theta_0} d\theta \frac{R^2 e^{2i\theta}}{\sqrt{R^2 e^{2i\theta} + m^2}} e^{iR\cos\theta|\vec x - \vec y|} e^{-R\sin\theta|\vec x - \vec y|}}_{= I_{R_1}}$$

$$\underbrace{+ i\int_{\theta_0}^{\pi} d\theta \frac{R^2 e^{2i\theta}}{\sqrt{R^2 e^{2i\theta} + m^2}} e^{iR\cos\theta|\vec x - \vec y|} e^{-R\sin\theta|\vec x - \vec y|}}_{= I_{R_2}}$$

$$\underbrace{+ i\int_R^m dk \frac{(ik+\varepsilon)\, e^{i(\varepsilon + ik)|\vec x - \vec y|}}{\sqrt{\varepsilon^2 - k^2 + 2ik\varepsilon + m^2}}}_{\mathcal{J}_1}$$

$$\underbrace{+ i\int_m^R dk \frac{(ik-\varepsilon)\, e^{-k|\vec x - \vec y| - i\varepsilon|\vec x - \vec y|}}{\sqrt{\varepsilon^2 - k^2 - 2ik\varepsilon + m^2}}}_{= \mathcal{J}_2}$$

$$\underbrace{+ i\int_0^\pi d\theta\, \varepsilon e^{i\theta} \frac{(im + \varepsilon e^{i\theta})\, e^{(im+\varepsilon e^{i\theta})i|\vec x - \vec y|}}{\sqrt{-m^2 + \varepsilon^2 e^{2i\theta} + 2im\varepsilon e^{i\theta} - m^2}}}_{= I_\varepsilon}$$

$$\int_{-\infty}^{\infty} dp \frac{p}{\sqrt{p^2+m^2}} e^{ip|\vec x - \vec y|} = \lim_{R\to\infty} \lim_{\varepsilon\to 0} \left(\int_C dz\, f(z) - I_{R_1} - I_{R_2} - \mathcal{J}_1 - \mathcal{J}_2 - I_\varepsilon\right)$$

$$\overset{*}{=} -i\int_\infty^m dk \frac{\cancel{i}k}{\cancel{i}\sqrt{k^2-m^2}} e^{-k|\vec x - \vec y|} - i\int_m^\infty dk \frac{\cancel{i}k\, e^{-k|\vec x - \vec y|}}{-\cancel{i}\sqrt{k^2-m^2}}$$

$$= 2i\int_m^\infty dk \frac{k\, e^{-k|\vec x - \vec y|}}{\sqrt{k^2-m^2}} = 2im\, K_1(m|\vec x - \vec y|)$$

The commutator is finally

$$[\phi(x), \phi^*(y)] = \frac{4\pi}{\cancel{4i}\, 2(2\pi)^3|\vec x - \vec y|} 2\cancel{i}m\, K_1(m|\vec x - \vec y|)$$

$$= \frac{m}{(2\pi)^2|\vec x - \vec y|} K_1(m|\vec x - \vec y|) \neq 0\ !$$

* Here we use $\lim_{R\to\infty} I_{R_1} = \lim_{R\to\infty} I_{R_2} = \lim_{\varepsilon\to 0} I_\varepsilon = 0$

(the integrands $Re^{-R\sin\theta|\vec x - \vec y|} \to 0$, $\sqrt{\varepsilon} \to 0$)

and $\lim_{\varepsilon\to 0} \sqrt{-k^2+m^2 \pm 2ik\varepsilon} = \pm i\sqrt{k^2-m^2}$.

b)

$$[\Phi(x), \Phi^*(y)] = [\phi(x) + \psi^*(x), \phi^*(y) + \psi(y)]$$

$$= [\phi(x), \phi^*(y)] + [\psi^*(x), \psi(y)] + \underbrace{[\phi(x), \psi(y)]}_{=0} + \underbrace{[\psi^*(x), \phi^*(y)]}_{=0}$$

(since $[c_p, a_q] = [c_p^*, a_q^*] = 0$

This is the same calculation as above with $\vec y \leftrightarrow \vec x$, and $m \to m_\psi$

$$[\psi^*(x), \psi(y)] = -[\psi(y), \psi^*(x)]$$

$$= -\frac{m_\psi}{(2\pi)^2|\vec y - \vec x|} K_1(m_\psi|\vec y - \vec x|)$$

$$[\Phi(x), \Phi^*(y)] = \frac{m}{(2\pi)^2|\vec x - \vec y|} K_1(m|\vec x - \vec y|) - \frac{m_\psi}{(2\pi)^2|\vec y - \vec x|} K_1(m_\psi|\vec y - \vec x|)$$

This clearly vanishes if $m = m_\psi$.

b) For fermions and spacelike intervals $(x-y)^2 > 0$:

$$\{\phi(x), \phi^*(y)\} = \int \frac{d^3p\, d^3q}{(2\pi)^3 2\sqrt{E_p E_q}} e^{i(px-qy)} \underbrace{\{a_{\vec p}, a_{\vec q}^*\}}_{= \delta^3(\vec p - \vec q)}$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_{\vec p}} e^{ip_\mu(x^\mu - y^\mu)}$$

$$= \frac{m}{(2\pi)^2|\vec x - \vec y|} K_1(m|\vec x - \vec y|)$$

$$\{\Phi(x), \Phi^*(y)\}$$

$$= \{\phi(x), \phi^*(y)\} + \{\psi^*(x), \psi(y)\} + \underbrace{\{\phi(x), \psi(y)\}}_{=0} + \underbrace{\{\psi^*(x), \phi^*(y)\}}_{=0}$$

$$= \frac{m}{(2\pi)^2|\vec x - \vec y|} K_1(m|\vec x - \vec y|) + \frac{m_\psi}{(2\pi)^2|\vec x - \vec y|} K_1(m_\psi|\vec x - \vec y|)$$

Since $m, m_\psi > 0$ this doesn't vanish.

$$[\phi(x), \phi^*(y)] = \{\phi(x), \phi^*(y)\} - 2\phi^*(y)\phi(x)$$

$$= \{\phi(x), \phi^*(y)\} - 2\int \frac{d^3p\, d^3q}{(2\pi)^3 2\sqrt{E_p E_q}} e^{i(px-qy)} a_q^* a_p$$

$$[\Phi(x), \Phi^*(y)] = \{\Phi(x), \Phi^*(y)\}$$

$$-2\int \frac{d^3p\, d^3q}{(2\pi)^3 2} \left(e^{-i(px-qy)} \frac{c_q c_p^*}{\sqrt{E_q E_p}} + e^{i(px-qy)} \frac{a_q^* a_p}{\sqrt{E_p E_q}} + e^{i(px+qy)} \frac{c_q a_p}{\sqrt{E_q E_p}} + e^{-i(qy+px)} \frac{a_q^* c_p^*}{\sqrt{E_q E_p}} \right)$$

$$\neq 0$$

No choice of m_ψ makes $[\Phi(x), \Phi^*(y)]$ or $\{\Phi(x), \Phi^*(y)\}$ vanish for $(x-y)^2 > 0$!