1 Exact Results

The first group of problems involve proving some exact results for QED, whose lagrangian density is given by $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm int}$, where

$$\mathcal{L}_0 = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} - \overline{\psi}(\partial\!\!\!/ + m)\psi \quad \text{and} \quad \mathcal{L}_{\rm int} = iq\,\overline{\psi}A\!\!\!/\psi\,, \tag{1.1}$$

as would be obtained by substituting $\partial_\mu \to D_\mu = \partial_\mu - iqA_\mu$ in $\mathcal{L}_0$. As usual $\partial\!\!\!/ := \gamma^\mu\partial_\mu$ and $A\!\!\!/ := \gamma^\mu A_\mu$ and $\overline{\psi} := \psi^\dagger\beta = i\psi^\dagger\gamma^0$.

The noninteracting field expansions appropriate to the interaction representation are

$$A_\mu(x) = \sum_{\lambda=\pm1}\int \frac{{\rm d}k}{\sqrt{(2\pi)^3 2\omega_k}}\left[\varepsilon_\mu(\mathbf{k}\lambda)\,\mathfrak{a}_{\mathbf{k},\lambda}\,e^{ikx} + \varepsilon^*_\mu(\mathbf{k}\lambda)\,\mathfrak{a}^*_{\mathbf{k},\lambda}\,e^{-ikx}\right]\,, \tag{1.2}$$

and

$$\psi(x) = \sum_{\sigma=\pm\frac12}\int \frac{{\rm d}p}{\sqrt{(2\pi)^3 2E_p}}\left[u(\mathbf{p},\sigma)\,\mathfrak{c}_{\mathbf{p}\sigma}\,e^{ipx} + v(\mathbf{p},\sigma)\,\overline{\mathfrak{c}}^*_{\mathbf{p}\sigma}\,e^{-ipx}\right]\,, \tag{1.3}$$

where $\mathfrak{a}_{\mathbf{k}\lambda}$ commutes with $\mathfrak{c}_{\mathbf{p}\sigma}$, $\overline{\mathfrak{c}}_{\mathbf{p}\sigma}$ and their adjoints while $\mathfrak{c}_{\mathbf{p}\sigma}$ anticommutes with $\mathfrak{c}_{\mathbf{q}\zeta}$, $\overline{\mathfrak{c}}_{\mathbf{q}\zeta}$ and $\overline{\mathfrak{c}}^*_{\mathbf{q}\zeta}$ (and the same with $\mathfrak{c} \leftrightarrow \overline{\mathfrak{c}}$) while

$$[\mathfrak{a}_{\mathbf{k}\lambda}, \mathfrak{a}^*_{\mathbf{q}\zeta}] = \delta_{\lambda\zeta}\,\delta^3(\mathbf{k}-\mathbf{q}) \quad \text{and} \quad \{\mathfrak{c}_{\mathbf{p}\sigma}, \mathfrak{c}^*_{\mathbf{q}\xi}\} = \{\overline{\mathfrak{c}}_{\mathbf{p}\sigma}, \overline{\mathfrak{c}}^*_{\mathbf{q}\xi}\} = \delta_{\sigma\xi}\,\delta^3(\mathbf{p}-\mathbf{q})\,, \tag{1.4}$$

and the polarization vectors $\varepsilon_\mu(\mathbf{k},\lambda)$, $u(\mathbf{p},\sigma)$ and $v(\mathbf{p},\sigma)$ are as given in the lectures. The polarization tensor satisfies $k^\mu\varepsilon_\mu(\mathbf{k},\lambda) = n^\mu\varepsilon_\mu(\mathbf{k},\lambda) = 0$ where $n^\mu = (1,0,0,0)$ points purely in the time direction (in the frame where $A_0 = \nabla\cdot\mathbf{A} = 0$).

1.1 Propagators and the Miracle of Lorentz Invariance

1. **Evaluate** the photon's Feynman propagator from scratch using the above field expansion, and prove the result shown in class:

$$\langle 0|T\Big[A_\mu(x)\,A_\nu(y)\Big]|0\rangle = -i\int\frac{{\rm d}^4p}{(2\pi)^4}\,\frac{\mathcal{S}_{\mu\nu}(p)}{p^2 - i\delta}\,e^{ip(x-y)} \tag{1.5}$$

   where δ is a positive infinitesimal and $p(x-y) := \eta_{\mu\nu}p^\mu(x-y)^\nu = -p^0(x-y)^0 + \mathbf{p}\cdot(\mathbf{x}-\mathbf{y})$ while

$$\begin{aligned}\mathcal{S}_{\mu\nu}(p) \;&:=\; \sum_{\lambda=\pm}\varepsilon_\mu(\mathbf{p},\lambda)\,\varepsilon^\dagger_\nu(\mathbf{p},\lambda)\\ &=\; \eta_{\mu\nu} - \frac{(n_\mu p_\nu + n_\nu p_\mu)p_0 + p_\mu p_\nu}{\mathbf{p}^2} + \frac{p^\lambda p_\lambda n_\mu n_\nu}{\mathbf{p}^2}\,.\end{aligned} \tag{1.6}$$

2. **Verify** explicitly that given the interaction hamiltonian density

$$\mathcal{H}_{\rm int}(\mathbf{x},t) = -J^\mu(\mathbf{x},t)A_\mu(\mathbf{x},t) + \frac{1}{8\pi}\int {\rm d}^3y\,\frac{J^0(\mathbf{x},t)J^0(\mathbf{y},t)}{|\mathbf{x}-\mathbf{y}|} \tag{1.7}$$

(where $J^\mu(x)$ is a classical current that satisfies $\partial_\mu J^\mu = 0$), the Lorentz noncovariant $(J^0)^2$ term cancels to second order in the S-matrix element

$$\langle 0|S|0\rangle = 1 - i\int \mathrm{d}^4x\; \langle 0|\mathcal{H}_{\rm int}(x)|0\rangle - \frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\; \langle 0|T\Big[\mathcal{H}_{\rm int}(x)\mathcal{H}_{\rm int}(y)\Big]|0\rangle + \cdots \tag{1.8}$$

3. **Evaluate** the electron's Feynman propagator from scratch using the above field expansion, and **prove** the result:

$$\langle 0|T\Big[\psi(x)\,\overline{\psi}(y)\Big]|0\rangle = -i\int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{\mathcal{S}(p)}{p^2+m^2-i\delta}\,e^{ip(x-y)} \tag{1.9}$$

where now

$$\mathcal{S}(p) := \sum_{\sigma=\pm\frac{1}{2}} u(\mathbf{p},\sigma)\,\overline{u}(\mathbf{p},\sigma) = -i\not{p} + m\,. \tag{1.10}$$

This also involves proving that $v(\mathbf{p},\sigma) = \gamma_5 u(\mathbf{p},-\sigma)$ satisfies

$$\sum_{\sigma=\pm\frac{1}{2}} v(\mathbf{p},\sigma)\,\overline{v}(\mathbf{p},\sigma) = -i\not{p} - m\,. \tag{1.11}$$

1.2 Low's Theorem

The relationship between Heisenberg-picture and Interaction-picture field operators is

$$\mathcal{O}_{H}(\mathbf{x},t) = \Omega(t)\mathcal{O}_{I}(\mathbf{x},t)\,\Omega^{-1}(t) \quad \text{where} \quad \Omega(t) := e^{iHt}\,e^{-iH_0t} \tag{1.12}$$

where the Hamiltonian is split between an unperturbed and perturbed part: $H = H_0 + H_{\rm int}$. Exact scattering states $|\alpha,t\rangle\!\rangle$ of the full system are given in terms of the corresponding unperturbed state $|\alpha\rangle$ by

$$|\alpha,t\rangle\!\rangle = \Omega(t)\,|\alpha\rangle \quad \text{so, in particular} \quad |\alpha\rangle\!\rangle_{\rm in} = \Omega(-\infty)\,|\alpha\rangle \quad \text{and} \quad |\alpha\rangle\!\rangle_{\rm out} = \Omega(\infty)\,|\alpha\rangle\,. \tag{1.13}$$

The evolution operator $U(t,t') := \Omega^{-1}(t)\Omega(t')$ satisfies the usual evolution equation

$$\frac{\partial U(t,t')}{\partial t} = -iV(t)\,U(t,t') \quad \text{where} \quad V(t) := \Omega^{-1}(t)H_{\rm int}\Omega(t)\,, \tag{1.14}$$

and so is given perturbatively by

$$U(t,t') = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t'}^{t}\mathrm{d}\tau_1\cdots\mathrm{d}\tau_n\,T\Big[V(\tau_1)\cdots V(\tau_n)\Big]\,. \tag{1.15}$$

1. **Prove** Low's theorem, which states that Heisenberg-picture and interaction-picture correlation functions are related by

$$\begin{aligned} &{}_{\rm out}\langle\!\langle\beta|T\Big[\mathcal{O}^{i_1}_{H}(x_1)\cdots\mathcal{O}^{i_n}_{H}(x_n)\Big]|\alpha\rangle\!\rangle_{\rm in} \\ &\qquad = \sum_{N=0}^{\infty}\frac{(-i)^N}{N!}\int_{-\infty}^{\infty}\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{N}\,\langle\beta|T\Big[\mathcal{O}^{i_1}_{I}(x_1)\cdots\mathcal{O}^{i_n}_{I}(x_n)V(\tau_1)\cdots V(\tau_{N})\Big]|\alpha\rangle\,. \end{aligned} \tag{1.16}$$

This theorem allows one to make the connection between exact Heisenberg-picture correlators and perturbative expressions in the interaction picture. The usual expression for the S-matrix

$$
{}_{\rm out}\langle\!\langle\beta|\alpha\rangle\!\rangle_{\rm in} = \langle\beta|S|\alpha\rangle = \sum_{N=0}^{\infty}\frac{(-i)^N}{N!}\int_{-\infty}^{\infty}{\rm d}\tau_1\cdots{\rm d}\tau_N\,\langle\beta|T\Big[V(\tau_1)\cdots V(\tau_N)\Big]|\alpha\rangle\,.
$$

is a special case of this result. Recall in particular that the free-field expansions given in (1.2) and (1.3) apply only to the interaction-picture field operators.

2. It is the Heisenberg-picture operators that satisfy

$$
\Big[P^\mu\,,\mathcal{O}_H(x)\Big] = i\partial^\mu\mathcal{O}_H(x) \tag{1.17}
$$

and not the interaction-picture operators (which are not translated in time by the full hamiltonian $H = P^0$). Use this to **prove**

$$
{}_{\rm out}\langle\!\langle\beta|T\Big[\mathcal{O}_H^{i_1}(x_1{+}x)\cdots\mathcal{O}_H^{i_n}(x_n{+}x)\Big]|\alpha\rangle\!\rangle_{\rm in} = e^{i(p_{(\alpha)}-p_{(\beta)})x}\,{}_{\rm out}\langle\!\langle\beta|T\Big[\mathcal{O}_H^{i_1}(x_1)\cdots\mathcal{O}_H^{i_n}(x_n)\Big]|\alpha\rangle\!\rangle_{\rm in}\,, \tag{1.18}
$$

provided $P^\mu|\alpha\rangle\!\rangle_{\rm in} = p^\mu_{(\alpha)}|\alpha\rangle\!\rangle_{\rm in}$ and $P^\mu|\beta\rangle\!\rangle_{\rm out} = p^\mu_{(\beta)}|\beta\rangle\!\rangle_{\rm out}$. In particular, if the ground state is translation invariant – *i.e.* $P^\mu|0\rangle\!\rangle_{\rm in} = P^\mu|0\rangle\!\rangle_{\rm out} = 0$ – then

$$
{}_{\rm out}\langle\!\langle 0|T\Big[\psi_n(x)\overline{\psi}_m(y)\Big]|0\rangle\!\rangle_{\rm in} = {}_{\rm out}\langle\!\langle 0|T\Big[\psi_n(x-y)\overline{\psi}_m(0)\Big]|0\rangle\!\rangle_{\rm in} \tag{1.19}
$$

must be a function only of $(x-y)^\mu$.

3. Similarly, conserved charges Q commute with charged fields to give

$$
\Big[Q\,,\mathcal{O}_H^i(x)\Big] = -q^i\mathcal{O}_H^i(x) \tag{1.20}
$$

where q^i is the charge of the field $\mathcal{O}_H^i$. If $Q|\alpha\rangle\!\rangle_{\rm in} = Q_\alpha|\alpha\rangle\!\rangle_{\rm in}$ and $Q|\beta\rangle\!\rangle_{\rm out} = Q_\beta|\beta\rangle\!\rangle_{\rm out}$ then **show** that

$$
{}_{\rm out}\langle\!\langle\beta|T\Big[\mathcal{O}_H^{i_1}(x_1)\cdots\mathcal{O}_H^{i_n}(x_n)\Big]|\alpha\rangle\!\rangle_{\rm in} \neq 0 \quad \text{implies} \quad Q_\beta = Q_\alpha - \sum_{k=1}^{n} q^{i_k}\,. \tag{1.21}
$$

1.3 QED Ward Identity

In QED consider the Heisenberg-picture correlator

$$
F^\mu_{mn}(x,y,z) := {}_{\rm out}\langle\!\langle 0|T\Big[\psi_n(x)\,J^\mu(z)\,\overline{\psi}_m(y)\Big]|0\rangle\!\rangle_{\rm in} \tag{1.22}
$$

where m and n are Dirac indices and $J^\mu = iq\overline{\psi}\gamma^\mu\psi$ is the electric current operator for a fermion with charge q. Recall that charge conservation implies the Heisenberg evolution of the operators ensures $\partial J^\mu(z)/\partial z^\mu = 0$ as an exact statement. Because of this only the time-ordering step functions contribute when F^μ_{mn} is differentiated with respect to z^μ.

1. Use the above observation and the current-field commutation relation – see eq. (1.20)

$$\Big[J^0(\mathbf{x},t)\,,\psi_n(\mathbf{y},t)\Big] = -q\,\delta^3(\mathbf{x}-\mathbf{y})\,\psi_n(\mathbf{y},t)\,, \tag{1.23}$$

   to **show** that

$$\begin{aligned}\frac{\partial F^\mu_{mn}}{\partial z^\mu} &= -q\delta^4(x-z)_{\rm out}\langle\!\langle 0|T\Big[\psi_n(x)\,\overline{\psi}_m(y)\Big]|0\rangle\!\rangle_{\rm in} \\ &\qquad +q\delta^4(y-z)_{\rm out}\langle\!\langle 0|T\Big[\psi_n(x)\,\overline{\psi}_m(y)\Big]|0\rangle\!\rangle_{\rm in}\,. \end{aligned}\tag{1.24}$$

2. **Prove** that translation invariance of the vacuum implies $F^\mu_{nm}(x,y,z)$ satisfies

$$\left(\frac{\partial}{\partial x^\mu}+\frac{\partial}{\partial y^\mu}+\frac{\partial}{\partial z^\mu}\right)F^\mu_{nm}(x,y,z)=0\,. \tag{1.25}$$

3. In momentum space define the quantity $\Gamma^\mu(p',p)$ from the relation

$$\int \mathrm{d}^4x\,\mathrm{d}^4y\,F^\mu_{nm}(x,y,0)\,e^{ipy-ip'x} =: -iq\,S_{nk}(p')\,\Gamma^\mu_{kl}(p',p)\,S_{lm}(p)\,, \tag{1.26}$$

   where there is an implied sum on the repeated Dirac indices k, l and

$$-iS_{nk}(p) := \int \mathrm{d}^4x\;{}_{\rm out}\langle\!\langle 0|T\Big[\psi_n(x)\,\overline{\psi}_k(0)\Big]|0\rangle\!\rangle_{\rm in}\,e^{-ipx}\,. \tag{1.27}$$

   Use (1.24) and (1.25) to **derive** the exact Dirac-matrix identity

$$(p-p')_\mu\Gamma^\mu(p',p) = i\Big[S^{-1}(p')-S^{-1}(p)\Big]\,. \tag{1.28}$$

   This is called a Ward identity because it is a relation amongst correlation functions that follows from current conservation (and so from a symmetry of the problem).

1.4 Symmetries and Ward Identities

Consider a generating functional for correlation functions for some operator $\mathcal{O}(x)$, defined by

$$W[J] = \sum_{n=0}^{\infty}\frac{1}{n!}\int \mathrm{d}^4x_1\cdots\mathrm{d}^4x_n J(x_1)\cdots J(x_n)\,\langle\!\langle 0|T\Big[\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\Big]|0\rangle\!\rangle\,. \tag{1.29}$$

This has the property that its functional derivatives with respect to $J(x)$ give the correlation functions for the Heisenberg-picture operator $\mathcal{O}(x)$, with

$$\left[\frac{\delta^n W}{\delta J(x_1)\cdots\delta J(x_n)}\right]_{J=0} = \langle\!\langle 0|T\Big[\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\Big]|0\rangle\!\rangle\,. \tag{1.30}$$

Suppose there is a transformation of the form $J \to \hat{J}(J,\theta) = J + \theta^a K_a(J) + \cdots$, for some arbitrary infinitesimal parameters θ^a, for which the generating functional is left unchanged. This implies in particular that

$$\int \mathrm{d}^4x \, \frac{\delta W}{\delta J(x)} K_a = 0 \tag{1.31}$$

is true for all $J(x)$ and all a. Because this is an identity for all J it remains true when multiply differentiated with respect to J. The result is a collection of relations amongst the correlation functions called Ward identities. For instance differentiating once leads to

$$\begin{aligned} 0 &= \int \mathrm{d}^4x \left[\frac{\delta^2 W}{\delta J(y)\delta J(x)} K_a(x) + \frac{\delta W}{\delta J(x)} \, \frac{\delta K_a(x)}{\delta J(y)} \right] \\ &= \int \mathrm{d}^4x \left[\langle\!\langle 0|T\Big[\mathcal{O}(y)\cdots\mathcal{O}(x)\Big]|0\rangle\!\rangle K_a(x) + \langle\!\langle 0|\mathcal{O}(x)|0\rangle\!\rangle \, \frac{\delta K_a(x)}{\delta J(y)} \right] \end{aligned} \tag{1.32}$$

and so on.

1. It is possible to define a generating functional, $\Gamma[a_\mu, \psi, \overline{\psi}]$, whose derivatives give 1-particle irreducible correlation functions, which is invariant under gauge transformations of the form $\delta a_\mu = \partial_\mu \zeta$, $\delta\psi = iq\zeta\psi$ and $\delta\overline{\psi} = -iq\zeta\overline{\psi}$. Use this gauge invariance to **derive** the Ward identity satisfied by the correlation function

$$\left[\frac{\delta^2 \Gamma}{\delta a_\mu(x) \delta a_\nu(y)} \right]_{a=\psi=0} . \tag{1.33}$$

   Similarly **derive** the Ward identity that relates the correlation functions

$$\left[\frac{\delta^2 \Gamma}{\delta\psi(x)\delta\overline{\psi}(y)} \right]_{a=\psi=0} \quad \text{and} \quad \left[\frac{\delta^3 \Gamma}{\delta\psi(x)\delta\overline{\psi}(y)\delta a_\mu(z)} \right]_{a=\psi=0} . \tag{1.34}$$

   How does this relate to (1.28)?

2 Perturbative Calculations

These problems involve more explicit perturbative calculations.

2.1 Dirac Identities

Consider the matrices defined in class

$$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} , \quad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \epsilon = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \tag{2.1}$$

where I is the 2-by-2 unit matrix and $\varepsilon = i\sigma_2$ is the 2-by-2 antisymmetric matrix (whose upper right element is $+1$). The matrix ϵ appears when writing the charge-conjugation matrix in terms of the other two $C = \gamma_5 \epsilon \beta$. Notice $\beta = \beta^* = \beta^\dagger = \beta^T = \beta^{-1}$ and $\gamma_5 = \gamma_5^* = \gamma_5^\dagger = \gamma_5^T = \gamma_5^{-1}$ while $\epsilon = \epsilon^* = -\epsilon^\dagger = -\epsilon^T = -\epsilon^{-1}$. Both β and γ_5 commute with

ϵ while β and γ_5 anticommute with one another (verify this!). The gamma matrices in this representation are defined by

$$\gamma_0 = -\gamma^0 = i\beta = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \gamma^k = \begin{pmatrix} 0 & -i\sigma^k \\ i\sigma^k & 0 \end{pmatrix} \tag{2.2}$$

where σ^k are the usual Pauli matrices.

1. **Prove** the identities

$$\{\gamma^\mu\,,\gamma^\nu\} = 2\eta^{\mu\nu}\,, \quad C = -\gamma^2\,, \quad \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\,\epsilon_{\mu\nu\lambda\rho}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho \tag{2.3}$$

where our conventions are $\eta^{\mu\nu} = \text{diag}(-,+,+,+)$ and $\epsilon^{0123} = +1$ (and so $\epsilon_{0123} = -1$). Use these to **prove**

$$\gamma_L\gamma_{\mu\nu} = -\frac{i}{2}\epsilon_{\mu\nu\lambda\rho}\gamma_L\gamma^{\lambda\rho} \quad \text{and} \quad \gamma_R\gamma_{\mu\nu} = +\frac{i}{2}\epsilon_{\mu\nu\lambda\rho}\gamma_R\gamma^{\lambda\rho} \tag{2.4}$$

where $\gamma_L := \frac{1}{2}(1+\gamma_5)$ and $\gamma_R := \frac{1}{2}(1-\gamma_5)$ and $\gamma_{\mu\nu} := \frac{1}{2}[\gamma_\mu, \gamma_\nu]$.

2. For the basis of Dirac matrices

$$M = \Big\{ I, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \gamma_{\mu\nu} \Big\} \tag{2.5}$$

compute the signs η, ζ and ξ that appear in the identities

$$M^T = \xi\,\epsilon M\epsilon\,, \quad M^\dagger = \zeta\,\beta M\beta\,, \quad M^* = \xi\zeta\,\epsilon\beta M\epsilon\beta\,, \quad \gamma_5 M\gamma_5 = \eta\,M \tag{2.6}$$

and **show** that they are given by the entries in the following table:

	I	γ_5	γ_μ	$\gamma_5\gamma_\mu$	$\gamma_{\mu\nu}$
ξ	$-$	$-$	$-$	$+$	$+$
ζ	$+$	$-$	$-$	$-$	$-$
η	$+$	$+$	$-$	$-$	$+$
$\xi\eta$	$-$	$-$	$+$	$-$	$+$
$\xi\zeta$	$-$	$+$	$+$	$-$	$-$
$\xi\zeta\eta$	$-$	$+$	$-$	$+$	$-$

Table 1: The signs relevant to Dirac matrix identities

These identities are useful when computing the reality and symmetry properties of fermion bilinears, such as when computing

$$(\overline{\psi}_1 M\psi_2)^* = (\psi_1^\dagger\beta M\psi_2)^* = \psi_2^\dagger M^\dagger\beta\psi_1) = \overline{\psi}_2\beta M^\dagger\beta\psi_1 = \zeta(\overline{\psi}_2 M\psi_1)\,. \tag{2.7}$$

3. **Prove** the following gamma-matrix trace identities

$$\text{Tr}\,(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) = \text{Tr}\,(\gamma_5\gamma^{\mu_1}\cdots\gamma^{\mu_n}) = 0 \quad \text{if } n \text{ is odd} \tag{2.8}$$

as well as $\text{Tr}\,(\gamma_5) = \text{Tr}\,(\gamma_5\gamma^\mu\gamma^\nu) = 0$. Also

$$\text{Tr}\;I = 4\,, \quad \text{Tr}\,(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu}\,, \quad \text{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho) = 4(\eta_{\mu\nu}\eta_{\lambda\rho} - \eta_{\mu\lambda}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\lambda}) \tag{2.9}$$

and

$$\text{Tr}\,(\gamma_5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho) = 4i\epsilon^{\mu\nu\lambda\rho}\,, \tag{2.10}$$

with the convention (as above) that $\epsilon^{0123} = +1$.

2.2 The Feynman Parameter Trick

1. Prove the very useful identity

$$\frac{1}{A_1 A_2 \cdots A_n} = (n-1)! \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{n-2}} \mathrm{d}x_{n-1} \Big[A_n x_{n-1} + A_{n-1}(x_{n-2} - x_{n-1}) + \cdots A_2(x_1 - x_2) + A_1(1 - x_1)\Big]^{-n}. \tag{2.11}$$

This contains the version used in class:

$$\frac{1}{A_1 A_2} = \int_0^1 \frac{\mathrm{d}x}{[A_1(1-x) + A_2 x]^2}\,. \tag{2.12}$$

2.3 Compton Scattering

Consider the scattering process

$$e^-(\mathbf{p}, \sigma) + \gamma(\mathbf{k}, \lambda) \to e^-(\mathbf{p}', \sigma') + \gamma(\mathbf{k}', \lambda') \tag{2.13}$$

where we denote the electron energies by $p^0 = \varepsilon = \sqrt{\mathbf{p}^2 + m^2}$ and $k^0 = \omega = |\mathbf{k}|$ and similarly for the final-state energies ε' and ω'.

Suppose the S matrix element for this process turns out to be given by

$$\mathcal{S} = \langle \mathbf{p}', \sigma'; \mathbf{k}', \lambda' | S | \mathbf{p}, \sigma; \mathbf{k}, \lambda \rangle = -2\pi i \delta^4(p + k - p' - k')\, \mathcal{M} \tag{2.14}$$

then the differential cross section is given in terms of $\mathcal{M}$ by

$$\mathrm{d}\sigma = \frac{(2\pi)^4}{u} |\mathcal{M}|^2 \delta^4(p + k - p' - k')\, \mathrm{d}^3k'\, \mathrm{d}^3p' \tag{2.15}$$

where

$$u := \frac{|p \cdot k|}{\varepsilon\omega}\,. \tag{2.16}$$

1. In the rest frame of the initial electron **show** that energy-momentum conservation implies

$$\omega' = \frac{\omega}{1 + (\omega/m)(1 - \cos\theta)} \tag{2.17}$$

where $\cos\theta := \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'$ is the angle between the incoming and outgoing photon directions. **Show** also that $u = 1$ and perform the integration over the outgoing electron momentum to **show** that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = (2\pi)^4 \frac{(\omega')^3(m + \omega - \omega')}{m\omega} |\mathcal{M}|^2\,, \tag{2.18}$$

where $\mathrm{d}^3k' = (k')^2 \mathrm{d}k' \mathrm{d}\Omega$ and $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ is the differential solid angle for the direction of the outgoing photon in a frame where the initial photon is moving up the positive z axis.

2. **Draw** the two distinct Feynman graphs that contribute to this process at lowest order in the electromagnetic coupling. Evaluate the graphs to **show** that

$$\mathcal{M} = \frac{q^2}{32\pi^3\sqrt{m\varepsilon'\omega\omega'}}\,(\varepsilon'_\mu)^*\varepsilon_\nu\overline{u}'\left[\gamma^\mu\frac{-i(\not{p}+\not{k})+m}{-2m\omega}\gamma^\nu+\gamma^\nu\frac{-i(\not{p}'-\not{k}')+m}{2m\omega'}\gamma^\mu\right]u \quad (2.19)$$

where $\varepsilon_\nu = \varepsilon(\mathbf{k},\lambda)$, $\varepsilon'_\mu = \varepsilon(\mathbf{k}',\lambda')$, $u = u(\mathbf{p},\sigma)$ and $u' = u(\mathbf{p}',\sigma')$.

If no spins or polarizations are measured we must sum over the final spins and average over the initial (unknown) spins, and so must evaluate

$$\overline{\mathcal{M}^2} = \frac{1}{4}\sum_{\lambda\lambda'\sigma\sigma'}|\mathcal{M}^2|\,. \quad (2.20)$$

Use (1.6), (1.10) and (1.11) to evaluate the spin sums and **show** that the unpolarized cross section can be written in the initial electron's rest frame as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{q^4}{32\pi^2m^2}\,\frac{1}{[1+(\omega/m)(1-\cos\theta)]^2}\left[1+\cos^2\theta+\frac{(\omega/m)^2(1-\cos\theta)^2}{1+(\omega/m)(1-\cos\theta)}\right]\,. \quad (2.21)$$

This reduces (as it must) to the Thompson cross section for $\omega \ll m$ and becomes sharply peaked in the forward direction when $\omega \gg m$. For $(\omega/m)(1-\cos\theta) \gg 1$ the approximate ultra-relativistic form is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \simeq \frac{q^4}{32\pi^2m\omega(1-\cos\theta)}\,. \quad (2.22)$$

Although this seems singular as $\theta \to 0$ the above expansion breaks down in the collinear regime where $1-\cos\theta \sim m/\omega \ll 1$, and for smaller scattering angles than this goes over to Thompson scattering again.

2.4 Mott Scattering

Mott scattering is the relativistic spin-half generalization of Rutherford scattering from a Coulomb potential. Consider an electron scattering $e^-(\mathbf{p},\sigma) \to e^-(\mathbf{p}',\sigma')$ from a fixed classical electrostatic field with $\mathbf{A} = 0$ and

$$\mathcal{A}^0(\mathbf{x}) = \frac{Ze}{4\pi|\mathbf{x}|}\,. \quad (2.23)$$

1. Compute the leading amplitude for scattering of an electron (with charge $q = -e$) from such a potential using the interaction lagrangian density

$$\mathcal{L}_{\rm int} = -ie\,\mathcal{A}^\mu\overline{\psi}\gamma_\mu\psi\,, \quad (2.24)$$

and **show** that the S-matrix element is given by

$$\langle\mathbf{p}',\sigma'|S|\mathbf{p},\sigma\rangle = i\int \mathrm{d}^4x\langle\mathbf{p}',\sigma'|\mathcal{L}_{\rm int}(x)|\mathbf{p},\sigma\rangle = -2\pi i\delta(\varepsilon-\varepsilon')\mathcal{T}\,, \quad (2.25)$$

with

$$\mathcal{T} = \frac{ie}{(2\pi)^3 2\varepsilon}\,\overline{u}'\gamma_0 u\int \mathrm{d}^3x\mathcal{A}^0(\mathbf{x})\,e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} = \frac{Ze^2}{(2\pi)^3 2\varepsilon}\,\frac{\overline{u}'\gamma_0 u}{|\mathbf{p}-\mathbf{p}'|^2}\,, \quad (2.26)$$

2. Writing the differential cross section as

$$d\sigma = \frac{(2\pi)^4}{v} |\mathcal{T}|^2 \delta(\varepsilon' - \varepsilon)\, d^3p' \tag{2.27}$$

show that the differential cross section (as a function of the final electron's scattering direction θ relative to the incoming electron direction) becomes

$$\frac{d\sigma}{d\Omega} = \frac{2(Z\alpha)^2}{|\mathbf{p}' - \mathbf{p}|^4} \Big(m^2 + \varepsilon^2 + |\mathbf{p}|^2 \cos\theta \Big) , \tag{2.28}$$

where $\alpha := e^2/(4\pi)$ is the usual fine-structure constant, and so in the ultra-relativistic limit $\varepsilon \simeq |\mathbf{p}| \gg m$ becomes

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z\alpha}{2\varepsilon} \right)^2 \frac{\cos^2 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2}} . \tag{2.29}$$

How does this result differ from Rutherford scattering? What is the physical reason why they should differ?

2.5 Bremstrahlung and Infrared Divergences

Next, compute the rate for the scattered electron to emit a photon during Mott scattering process, so $e^-(\mathbf{p}, \sigma) \to e^-(\mathbf{p}', \sigma') + \gamma(\mathbf{k}, \lambda)$ in the presence of the classical Coulomb field (2.23) (with $\mathbf{A}_{\text{class}} = 0$).

1. First draw the two relevant Feynman graphs for this process, and show that they imply the Mott scattering matrix element (2.26) is modified by replacing $\overline{u}'\gamma_0 u$ with

$$\left(\frac{-ie\,\varepsilon^*_\mu}{\sqrt{(2\pi)^3 2\omega}} \right) \overline{u}' \left\{ \gamma^\mu \left[\frac{-i(\not{p}' + \not{k}) + m}{(p' + k)^2 + m^2 - i\delta} \right] \gamma_0 + \gamma_0 \left[\frac{-i(\not{p} - \not{k}) + m}{(p - k)^2 + m^2 - i\delta} \right] \gamma^\mu \right\} u \tag{2.30}$$

where $\varepsilon_\mu = \varepsilon_\mu(\mathbf{k}, \lambda)$ and so on.

2. Modifying (2.27) to sum over the final-state photon momentum gives

$$d\sigma = \frac{(2\pi)^4}{v} |\mathcal{T}|^2 \delta(\varepsilon' + \omega - \varepsilon)\, d^3p'\, d^3k \tag{2.31}$$

where $\omega := k^0 = |\mathbf{k}|$ is the final-state photon energy. Use this to **show** that the total integrated cross section σ diverges once the d^3k integration is performed, due to the infrared region $|\mathbf{k}| \to 0$. Called the 'infrared catastrophe', this reflects the great ease with which charged systems can radiate very soft photons.

3. The differential photon emission rate is suppressed by a factor of $\alpha = e^2/4\pi$ relative to the leading Mott cross section given in (2.28). But at this order one should also include the modification to the amplitude for Mott scattering (without the emission of an extra

photon) by the exchange of a photon between the initial and final electron. **Show** that the scattering matrix element $\mathcal{T}$ for this is obtained from (2.26) by replacing $\overline{u}'\gamma_0 u$ by

$$\overline{u}'\gamma_0 u + ie^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 - i\delta} \Big(\eta_{\mu\nu} + k_\mu \alpha_\nu + k_\nu \alpha_\mu\Big) \times \overline{u}' \left\{ \gamma^\mu \left[\frac{-i(\not{p}' - \not{k}) + m}{(p'-k)^2 + m^2 - i\delta} \right] \gamma_0 \left[\frac{-i(\not{p} - \not{k}) + m}{(p-k)^2 + m^2 - i\delta} \right] \gamma^\nu \right\} u \tag{2.32}$$

which drops the $n_\mu n_\nu$ term of (1.6) because we work with the Path Integral formulation where the interaction is given by $\mathcal{L}_{\rm int}$ rather than $\mathcal{H}_{\rm int}$.

4. **Show** that the detailed form of α_μ does not matter because the terms involving them contribute zero to the amplitude $\mathcal{T}$.

5. **Show** that the interference cross term between the $\overline{u}'\gamma_0 u$ term and the rest that arises within $|\mathcal{T}|^2$ involves a single integration over k^μ that also diverges in the infrared (from the integration limit $|\mathbf{k}| \to 0$). Show that this divergence has a coefficient that precisely cancels the divergence in photon emission if one were to add the rate for Mott scattering to the rate for scattering accompanied by the the emission of one photon (as would be appropriate if we work to a consistent order in powers of α and the emitted photon's energy is so small that it is not detected).