Thermal Instability (Fields, 1965)

- consider infinite static medium of density ρ_0, temperature T_0
  (→ continuity + force-balance automatically satisfied)
- let $\mathcal{L}(\rho_0, T_0)$ = energy losses − energy gains per gram per sec

not used fall 2006 or fall 2010

$$= \frac{\Lambda - \Gamma}{\rho}$$

$\mathcal{L}$ is generalized heat loss function → determines T_0 for any given ρ_0

Stability: Introduce perturbation for which T and ρ are perturbed, keeping some other thermodynamic variable A constant (ie pressure)

- since $dQ = T\,dS$, change in entropy δS occurs while heat-loss function undergoes a change $\delta\mathcal{L}$ (both calculated at constant A)

(δ represents difference between parcel & surrounding)

So $\boxed{\delta \mathcal{L}\, dt = -T\, d(\delta S)}$

Instability occurs if change in $\mathcal{L}$ has sign <u>opposite</u> to change in entropy

$$\boxed{\left(\frac{\partial \mathcal{L}}{\partial S}\right)_A < 0}$$

Isochoric : $T\,dS = C_v\,dT$ → not physical → P variations → motions → ρ variations

<u>Isobaric</u> : $T\,dS = C_p\,dT$

$$\left(\frac{\partial \mathcal{L}}{\partial T}\right)_p = \left(\frac{\partial \mathcal{L}}{\partial T}\right)_\rho - \frac{\rho_0}{T_0}\left(\frac{\partial \mathcal{L}}{\partial \rho}\right)_T < 0$$

Isentropic :

$$\left(\frac{\partial \mathcal{L}}{\partial T}\right)_S = \left(\frac{\partial \mathcal{L}}{\partial T}\right)_\rho + \frac{1}{(\gamma-1)}\frac{\rho_0}{T_0}\left(\frac{\partial \mathcal{L}}{\partial \rho}\right)_T < 0$$

$\left(\frac{\partial \mathcal{L}}{\partial \rho}\right)_T$ is often positive in astronomical applications (binary-collision nature of radiative losses)

__How to proceed__: Equations of concern ~~are~~ continuity, motion, energy, equation of st[ate]

$$\frac{d\rho}{dt} + \rho \nabla \cdot \bar{v} = 0$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \bar{v} \cdot \nabla$$

$$\rho \frac{d\bar{v}}{dt} + \nabla p = 0$$

$$p = \frac{R}{\mu} \rho T$$

$$\underbrace{\frac{1}{\gamma - 1} \frac{dp}{dt} - \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \frac{d\rho}{dt}}_{\rho T \left(\frac{ds}{dt}\right) \text{ for perfect gas}} + \rho \mathcal{L} = 0$$

Fields includes thermal conductivity

Equilibrium:

$$\rho = \rho_0 \quad T = T_0 \quad \bar{v} = 0 \quad \mathcal{L}(\rho_0, T_0) = 0$$

$\Rightarrow$ perturbations

$$\psi = \psi_1 \exp(\omega t + i \bar{k} \cdot \bar{r})$$

Equations governing perturbations:

$$\omega \rho_1 + \rho_0 \, i \bar{k} \cdot \bar{v}_1 = 0$$

$$\omega \rho_0 \bar{v}_1 + i \bar{k} \, p_1 = 0 \quad \Longrightarrow \quad \bar{k} \times \bar{v}_1 = 0$$

$$\frac{\omega}{\gamma - 1} p_1 - \frac{\omega \gamma p_0}{(\gamma - 1)\rho_0} \rho_1 + \rho_0 \underbrace{\left(\mathcal{L}_\rho \rho_1 + \mathcal{L}_T T_1 \right)}_{*} = 0$$

$$\frac{p_1}{p_0} - \frac{\rho_1}{\rho_0} - \frac{T_1}{T_0} = 0$$

$$* \quad \mathcal{L}_T = \left(\frac{\partial \mathcal{L}}{\partial T} \right)_\rho \Big|_{\rho_0} \qquad \mathcal{L}_\rho = \left(\frac{\partial \mathcal{L}}{\partial \rho} \right)_T \Big|_{T_0}$$

$\Rightarrow$ 4 equations in 4 unknowns

$\rightarrow$ vanishing determinant gives dispersion rel'n

$$\boxed{\omega^3 + c k_T \omega^2 + c^2 k^2 \omega + \frac{c^3 k^2}{\gamma} (k_T - k_\rho) = 0}$$

$$k_\rho = \frac{\mu (\gamma - 1) \rho_0 \mathcal{L}_\rho}{R c T_0} \qquad k_T = \frac{\mu (\gamma - 1) \mathcal{L}_T}{R c}$$

$$c = \left(\frac{\gamma p_0}{\rho_0} \right)^{1/2} = \text{speed of sound}$$

(we are ignoring conductivity)

→ look for <u>positive roots</u> → occur when

$k_T - k_g < 0$ or

$$\boxed{T_0 \, h_T - g_0 \, h_g < 0}$$

↑ This is the isobaric instability condition

⇒ Fields, Goldsmith, & Habing 1969

Spitzer, Ch 6

$$\underbrace{\overset{\text{thermal energy}}{\frac{3}{2}k\frac{dT}{dt}} - \overset{\text{work done}}{\frac{kT}{n}\frac{dn}{dt}}}_{T\frac{dS}{dt}} = \overset{\text{heating - cooling}}{\frac{\Gamma - \Lambda}{n}}$$

if n (or ρ) is constant (Field → "isochoric")

$$T\frac{dS}{dt} = \frac{3}{2}k\frac{dT}{dt}$$ ⇝ increase in entropy implies increase in thermal energy

if when you increase entropy it causes an increase in the net heating, the situation will be thermally unstable

increase in net heating = decrease in net cooling

Fields: $\mathcal{L}$ = "heat loss function"

via Spitzer → $T\frac{dS}{dt} = \frac{\Gamma - \Lambda}{n} = -\frac{(\Lambda - \Gamma)}{n} = -\mathcal{L}$ or $\mathcal{L}dt = TdS$ (Fields #2)

so will be unstable if increase in S gives decrease in $\mathcal{L}$, or

$$\left(\frac{d\mathcal{L}}{dS}\right)_\rho < 0$$

∴ I conclude Fields eq. 3 has a sign error